Discrete Mappings

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Abstract. Piecewise-linear triangle meshes are widely popular for surface representation in the digital computer; mappings of meshes are therefore central for applications such as computing good coordinate systems on surfaces (parameterization), finding correspondence between shapes, and physical simulation. However, representation and calculation of mappings in a computer pose several challenges: (i) how to define faithful discrete analogous of properties of smooth mappings (e.g., angle or area preservation), (ii) how to guarantee properties such as injectivity and/or surjectivity, and (iii) how to construct mappings between non-Euclidean (curved) domains. One particularly interesting sub-class of simplicial mappings is the collection of convex combination mappings, in which the image of each vertex of the triangulation is restricted to the convex-hull of its immediate neighbors' images. Convex combination mappings can guarantee injectivity, are simple to compute algorithmically, offer a discrete analog to harmonic mappings, and can be used to approximate conformal mappings. This lecture provides an introduction to convex combination mappings and their generalizations, as well as their algorithmic aspects and practical applications.

1. Triangulations and discrete mappings.

Surfaces in a computer are often represented or approximated by triangulations $M = (V, E, F)$, where $V = \{v_i\} \subset \mathbb{R}^d$ (typically $d = 2, 3$) is the vertex set, $E = \{e_{ij}\}$ the edge set, and $F = \{f_{ijk}\}$ the face set. Edges are convex-hulls of pairs of points, $e_{ij} = \text{hull}\{v_i, v_j\}$, and faces are convex-hulls of triplets of points $f_{ijk} = \text{hull}\{v_i, v_j, v_k\}$. M is a simplicial complex, meaning that:

(i) The intersection of any two faces is either an edge, a vertex or an empty set; the intersection of any two edges is either a vertex or empty.
(ii) All edges of a triangle are in E; all vertices of an edge are in V.

Note that a triangulation M is not necessarily a topological surface (*i.e.*, each point has a neighborhood homeomorphic to a disk), as shown for example in Figure 1(a). To assure a triangulation is indeed a topological surface we add the requirement that

(iii) The link of each vertex is a simple closed polygon.

The link of a vertex v_i is the union of all edges e_{jk} that do not contain v_i but share a triangle f_{ijk} with it, namely

$$\text{link}(v_i) = \bigcup_{\{e_{jk} \mid f_{ijk} \in F\}} e_{jk}.$$

Definition 1. *A* surface triangulation *is a triangulation* $M = (V, E, F)$ *satisfying* (i)-(iii).

For example, Figure 1(a)-left is not a surface triangulation since the link of the middle vertex consists of two closed polygonal loop; the right example is not a surface since the link of a vertex at any of the two ends of the center edge contains a subset of edges homeomorphic to the symbol Y.

To allow for surface triangulations with boundary we relax Condition (iii):

(iii') The link of each vertex is a simple closed polygon or a simple polygonal arc.

Definition 2. *A* triangulation of a surface with a boundary *is a triangulation* $M = (V, E, F)$ *satisfying* (i)-(ii) *and* (iii').

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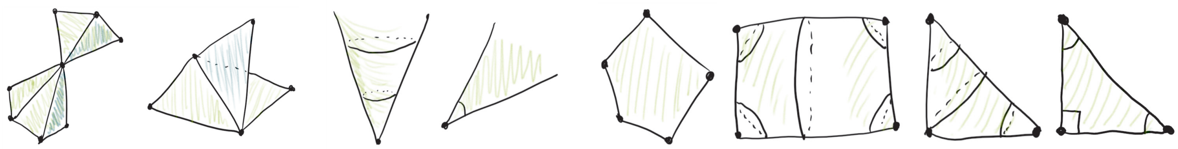


FIGURE 1. (a) Non manifold triangulations; (b) a cone (left) and a sector (right); (c) Euclidean cone surfaces: convex polygonal region (left), Euclidean orbifolds (middle, right).

The boundary of a surface triangulation M is a one-dimensional simplicial complex, that is a polygonal curve. It consists of all edges with only one adjacent face and all vertices whose link is a polygonal arc. We denote it by $M_B = (V_B, E_B)$. The interior vertex set is denoted $V_I = V \setminus V_B$. Note that in general, in contrast to our intuition from the continuous case, not every boundary vertex has an interior vertex as a neighbor; *e.g.*, stitch a triangle to the boundary of a disk-type mesh and consider its dangling vertex.

A surface triangulation $M = (V, E, F)$ is connected if its underlying graph $V = (V, E)$ is connected. It is 3-connected if it cannot be disconnected by removing any two vertices. If M is a boundaryless surface triangulation it is automatically 3-connected. If M is disk-type and 3-connected the relation of boundary vertices V_B to interior vertices B_I is similar to the continuous case: every boundary vertex has an interior vertex neighbor (if interior vertices exist). In fact, a stronger claim holds in the 3-connected case [**Flo03a**]: any interior vertex v_i can be connected to any other vertex v by a path $P = [v_i, v_1, v_2, \ldots, v_k, v]$ of interior vertices $v_j \in V_I$, $1 \le j \le k$.

We want to discuss mappings of surface triangulations. The most natural class of mappings of a triangulation consists of simplicial maps:

DEFINITION 3. *A* simplicial map $f : M \to \mathbb{R}^d$ *is a continuous piecewise-linear mapping defined as the unique piecewise-linear extension of a vertex map taking vertices $v_i \in V$ to $u_i \in \mathbb{R}^d$. By linear extension we mean that every point in the interior of some simplex σ (edge or face) $x = \sum_i \lambda_i v_i$, where $\lambda_i \ge 0$, $\sum_i \lambda_i = 1$, and v_i are the vertices of σ, is mapped to $f(x) = \sum_i \lambda_i u_i$.*

A guiding problem for us is:

PROBLEM 1. *Given two topologically equivalent surfaces*[1] M_1, M_2, *compute a simplicial homeomorphism* $f : M_1 \to M_2$. *We would like this homeomorphism to have some minimal distortion property and potentially interpolate given point data* $\{(x_i, y_i)\}_{i \in I} \subset M_1 \times M_2$.

Unfortunately, directly computing a homeomorphic simplicial map $M_1 \to M_2$ is in general a non-convex and difficult problem. We will tackle this problem by considering a canonical (topologically equivalent) domain $\mathcal{N}$ and show how to compute simplicial homeomorphisms $f_1 : M_1 \to \mathcal{N}$ and $f_2 : M_2 \to \mathcal{N}$. We then define $f = f_2^{-1} \circ f_1 : M_1 \to M_2$. The canonical surface $\mathcal{N}$ will be chosen from a particular family of special surfaces $\mathcal{F} = \{\mathcal{N}\}$ we will consider in this chapter.

2. Convex combination mappings.

Our goal is to build a simplicial homeomorphism $M \to \mathcal{N}$ onto a member surface of a special family of surfaces $\mathcal{F} = \{\mathcal{N}\}$. Building the homeomorphism $M \to \mathcal{N}$ will be accomplished by solving a system of sparse linear equations. The method of constructing this simplicial map is called *convex combination mapping.* It originated in [**Tut63**] and was generalized and given this name in [**Flo97, Flo03a, RG06**]. These works allowed homeomorphic simplicial mappings of topological disks onto convex polygonal regions. In [**Lov04, SF04, GGT06**] the embedding was generalized to the topological torus. In [**GGT06**] sufficient conditions for non-convex, as well as multiply connected domains were formulated. In [**AL15**], the construction was generalized to the collection of Euclidean orbifolds (which contains the torus as a particular case). The following is based mostly on the above papers.

$\mathcal{F}$ will soon be defined as a certain collection of Euclidean cone surfaces.

DEFINITION 4. *A compact oriented surface $\mathcal{N}$ is a* Euclidean cone surface *if it is a metric space locally isometric to an open disk, a cone, or a sector, and the number of cone points is finite.*

[1]Two surfaces are topologically equivalent if there exists a homeomorphism between them. Intuitively, the surfaces can be stretched onto one another without the need to tear the surfaces.

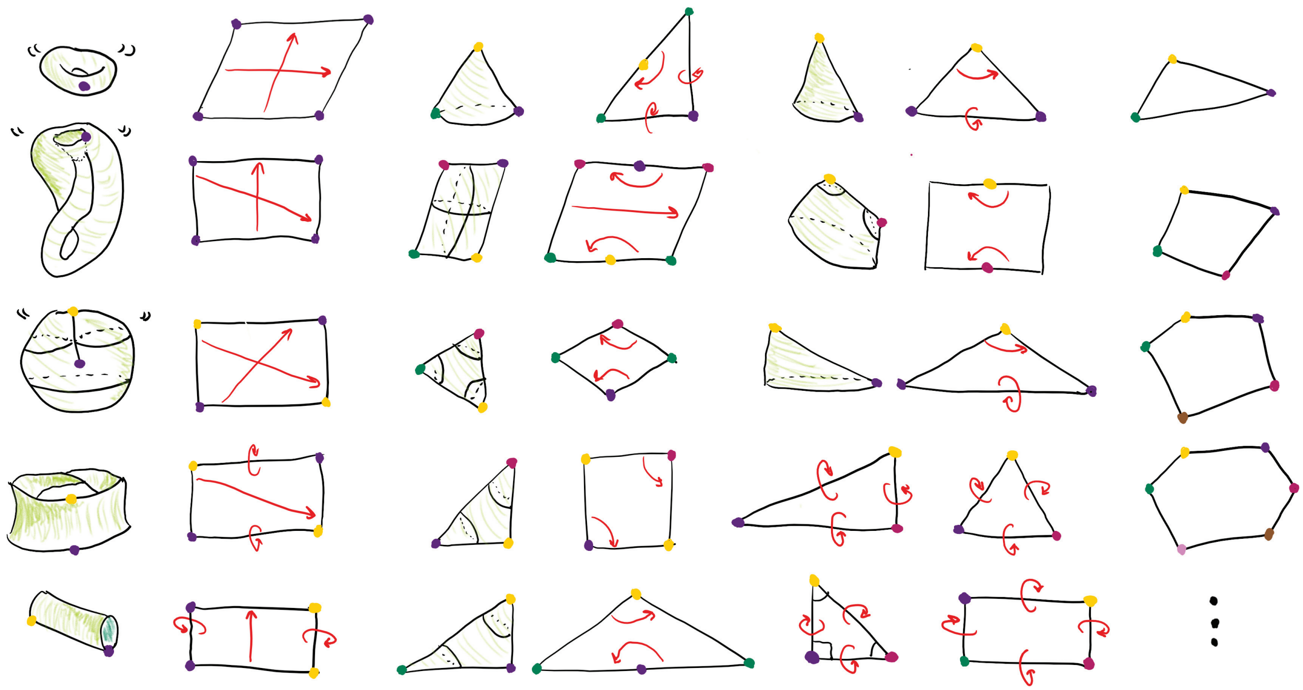


FIGURE 2. All types of fundamental domains; rigid motion identifications of edges are depicted with arrows.

A cone and a sector are shown in Figure 1(b). For each point $x \in \mathcal{N}$ we define the angle $\theta(x)$ to be the angle sum at the point x. Interior points have $\theta(x) = 2\pi$ and boundary points $\theta(x) = \pi$. An interior point is a cone point if $\theta(x) \neq 2\pi$. A boundary point is a cone point if $\theta(x) \neq \pi$. We will consider a specific family of Euclidean cone surfaces:

$$\mathcal{F} = \{\mathcal{N} \mid \mathcal{N} \text{ is a convex polygonal domain or a Euclidean orbifold}\}.$$

A convex polygonal domain is shown in Figure 1(c), left. Figure 1(c), right, shows three examples of Euclidean orbifolds: two topological spheres and a topological disk. Euclidean orbifolds is a specific family of Euclidean cone surfaces defined by taking the quotient of the plane $\mathbb{R}^2$ w.r.t. a wallpaper symmetry group of the plane, G, *i.e.*, $\mathbb{R}^2/G$. Put differently, a symmetry group G defines an equivalence relation $u \sim w$ iff $u = g(w)$ for some $g \in G$. Taking the quotient topology w.r.t. such an equivalence relation leads to a surface $\mathcal{N}$ where the orbits of G, namely $[u] = \{g(u) \mid g \in G\}$, are identified as points. The orbifolds are in one-to-one correspondence with the 17 wallpaper groups. Euclidean orbifolds are characterized by their topological type, and number, order and types of cones. There are two types of cones: reflective (where the surface is locally isometric to a sector, see Figure 1(b), right), and rotational (where the surface is locally isometric to a cone, see Figure 1(b), left). The order of a cone point x is $2\pi/\theta(x)$ for rotational and $\pi/\theta(x)$ for reflectional cones. Figure 3 lists all Euclidean orbifolds using the so-called orbifold notation [**CBGS08**]. Figure 1(c) shows (2222), (244), and (∗244) orbifolds, respectively.

Each Euclidean orbifold has a (non-unique) fundamental domain, namely a domain that represents a connected choice of a representative per orbit. As fundamental domains we use (closed) disk-type polygonal domains Ω defined by a closed convex polygonal curve $[p_1, p_2, \ldots, p_m]$ together with a list of identifications of pairs of edges $[p_i, p_{i+1}] \leftrightarrow [p_j, p_{j+1}]$ by rigid motions $g \in G$, *i.e.*, $g([p_i, p_{i+1}]) = [p_j, p_{j+1}]$. Note that these rigid identifications g also generate G. Pairs of different edges are identified with rotations, translations, or glides (a composition of a reflection and translation along a line), while self identified edges are identified with reflections. Figure 2 shows all fundamental domains where the identification are visualized by arrows. Arrows between different edges include rotations, translations, and glides (slanted arrows); self arrows indicate reflection across the infinite line supporting the edge. The edge identification implies an equivalence relation in $\Omega \times \Omega$ where identified points are in the same equivalence class. Once we quotient Ω by this relation (*i.e.*, the edges of Ω are stitched together) we achieve the orbifold $\mathcal{N}$. For convex polygonal domains we denote $\Omega = \mathcal{N}$ with all edges self-identified (right column in Figure 2).

topology	cones
sphere	(236), (244), (333), (2222)
disk	$(*236)$, $(*244)$, $(*333)$, $(*2222)$, $(2*22)$, $(3*3)$, $(4*2)$, $(22*)$
projective plane	$(22x)$
torus	(o)
Klein bottle	(xx)
annulus	$(**)$
Möbius band	$(*x)$

FIGURE 3. The types of Euclidean orbifolds. On left to $*$ (or without any $*$) are rotational cone orders; on the right to $*$ reflective cone orders.

We will build a simplicial mapping $f : M \to \mathcal{N}$ called a convex combination mapping. This is done by assigning a positive weight, $\omega_{ij} > 0$, to each edge $e_{ij} \in E$.

DEFINITION 5. *A* convex combination mapping $f : M \to \mathbb{R}^2$ *is a simplicial map, mapping each interior vertex* $v_i \in V_I$ *to a planar point* $u_i \in \mathbb{R}^2$ *so that*

$$\sum_{j \in \mathfrak{N}_i} \omega_{ij}(u_j - u_i) = 0, \tag{1}$$

where $\mathfrak{N}_i = \{j \mid e_{ij} \in E\}$ *is the neighbor index set of* v_i*. Geometrically,* u_i *is strictly inside the convex-hull defined by its immediate neighbors.*

An important property of convex combination mappings is the discrete maximum principle. It is useful to formulate it in the functional setting: Let $h : M \to \mathbb{R}$ satisfy the convex combination property (1) at each interior vertex V_I. That is, to each vertex $v_i \in M$ one associates a scalar $h_i \in \mathbb{R}$ and these scalars satisfy (1) for all $v_i \in V_I$. For example, one can consider the x- or y-coordinate of convex combination mapping. Then, as proved *e.g.*, in [**Flo03a**],

PROPOSITION 1. *(Discrete maximum principle.) Let* $h : M \to \mathbb{R}$ *be a convex combination function, and* M *a 3-connected disk-type surface triangulation. Let* $v_i \in V_I$*. If* $h_i = \min_j h_j$ *or* $h_i = \max_j h_j$ *then* h *is a constant function. In particular this implies that the maximum and minimum is achieved on the boundary* V_B*. The last assertion is true also if* M *is not 3-connected.*

PROOF. Assume $h_i = \min h_j$. The convex combination property together with the fact that h_i achieves the minimum imply that its immediate neighbors also equal h_i. Using the 3-connectedness we can construct an interior path to any other vertex, therefore continuing in this manner the proposition is proved. □

Convex combination mappings by themselves are not sufficient for building homeomorphisms. For example, the trivial (*i.e.*, constant) convex combination mapping $u_i = u$, $\forall i$ always exists. To achieve a homemorphism certain boundary conditions should be applied. These boundary conditions in essence force the image of the map, $f(M)$, to cover $\mathcal{N}$. Bijectivity follows from the particular properties of the family of surfaces $\mathcal{F}$ and is not true in general for all Euclidean cone surfaces.

We deal with M that has one of the topological types that appear in $\mathcal{F}$ (all possible topological types and fundamental domains are listed in Figure 3 and depicted in Figure 2). Given a choice of target cone surface $\mathcal{N} \in \mathcal{F}$ with m cone points, let its fundamental disk domain be denoted by Ω. Choose a simple connected polygonal path $\Gamma \subset V \cup E$ passing through at-least m vertices of M, denoted $C = \{v_{c_1}, v_{c_2}, \dots, v_{c_m}\} \subset V$, so that $M \setminus \Gamma$ is homeomorphic to Ω° (*i.e.*, the interior of Ω). Let $M' = \{V', E', F'\}$ be the triangulation representing M cut along Γ, where $F' = F$ and M' is homeomorphic to Ω. Figure 4 shows an example of (from left to right): triangulations M, M', Ω and a target orbifold $\mathcal{N}$.

We will need to relate the vertices, edges, and faces of M' and M. For that end we consider the inclusion map $\iota : M' \hookrightarrow M$. The inclusion ι induces an equivalence relation in M': $x \sim y \iff \iota(x) = \iota(y)$. Taking the quotient of M' with this equivalence relation results in M; this is equivalent to stitching M' along Γ to retrieve back M.

To compute the simplicial homeomorphism $f : M \to \mathcal{N}$ we consider M' and define a simplicial map $s : M' \to \mathbb{R}^2$ by solving a linear system of equations. The unknowns of the linear system are the target locations of the vertices in the plane, namely $u_i = s(v_i)$, $v_i \in V'$; the simplicial map s is the unique piecewise linear extension of this vertex map. The total degrees of freedom of s are therefore $u = \left\{u_1, u_2, \dots, u_{|V'|}\right\} \in \mathbb{R}^{2\times|V'|}$. The linear system consists of three sets of equations:

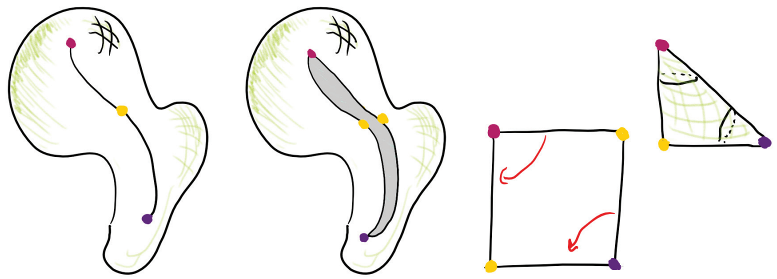


FIGURE 4. A surface triangulation M (left) is cut into M' (middle-left) to be homeomorphic to the fundamental domain Ω (middle-right) of a selected target $\mathcal{N}$ (right).

(a) For interior vertices, V_I', we set the convex combination equation (1), that is, every interior vertex is mapped strictly inside the convex hull of its immediate neighbors.
(b) For (non-cone) boundary vertices, $V_B' \setminus C'$, where $C' = \iota^{-1}(C) \subset V'$, we formulate equations preserving the edge identification rigid motions (see Figure 2), and maintaining the convex combination condition across the boundary of M'; Equations (3),(4),(5),(6).
(c) For cone vertices C', we restrict their position to the vertices of the fundamental domain Ω,

$$u_i = p_i, \quad v_i \in C'. \tag{2}$$

Equations (b) are dependent on the choice of target surface $\mathcal{N} \in \mathcal{F}$ and realize the edge identification in the fundamental domain. The equivalence relation $\sim$ induces an equivalence relation in the vertex set, $v_i \sim v_j$ in V'. The equivalence classes, $\langle v_i \rangle = \{v_{i'} \in V' \mid v_{i'} \sim v_i\}$, include interior vertices as singletons, and boundary vertices either as singletons, $V_{BS}' = \{\langle v_i \rangle \mid v_i \in V_B', \langle v_i \rangle = \{v_i\}\}$ (in case the point v_i is also boundary of M) or pairs $V_{BP}' = \{\langle v_i \rangle \mid v_i \in V_B', \langle v_i \rangle = \{v_i, v_{i'}\}\}$ (v_i is not a boundary of M). Singleton boundary vertices V_{BS}' are aligned by reflections, namely, stay on some infinite line,

$$a_i^T u_i + b_i = 0, \quad \forall \langle v_i \rangle \in V_{BS}', \tag{3}$$

where $0 \neq a_i \in \mathbb{R}^2$, $b_i \in \mathbb{R}$. Pairs of boundary vertices V_{BP}' are aligned by rotations, translations, or glide-reflections

$$u_i = r_{ii'} u_{i'} + t_{ii'}, \quad \forall \langle v_i \rangle \in V_{BP}', \tag{4}$$

where $r_{ii'} \in O(2)$ and $t_{ii'} \in \mathbb{R}^2$. Note that (3) and (4) do not capture all the degrees of freedom for non-cone boundary vertices, $V_B' \setminus C'$; there is still one degree of freedom for singleton boundary vertex and two degrees of freedom for a pair of boundary vertices. These degrees of freedom are used to assure the convex combination property is preserved across the boundary of M':

$$\sum_{j \in \mathfrak{N}_i} \omega_{ij} (a_i^{\perp})^T (u_j - u_i) = 0, \quad \forall \langle v_i \rangle \in V_{BS}', \tag{5}$$

where $a_i^{\perp} \perp a_i$, and

$$\sum_{j \in \mathfrak{N}_i} \omega_{ij} (u_j - u_i) + \sum_{j \in \mathfrak{N}_{i'}} \omega_{i'j} r_{ii'} (u_j - u_{i'}) = 0, \quad \forall \langle v_i \rangle \in V_{BP}'. \tag{6}$$

We denote the linear system

$$\mathcal{L} = (a) + (b) + (c).$$

The simplicial map $s : M' \to \mathbb{R}^2$ is defined by the solution of this equation (existence and uniqueness proof is deferred a bit). The simplicial map $s : M' \to \Omega$ defines a unique map $f : M \to \mathcal{N}$ by

$$f(\langle x \rangle) = [s(x)], \quad \forall x \in M'.$$

Remember that $\langle x \rangle$ represents a point in M. The map is well defined since pairs $y, z \in \langle x \rangle$ are in M_B' (*i.e.*, the boundary of M') and are mapped, due to the boundary conditions (3), (4), to points $f(y), f(z)$ in the same orbit of G, that is, $[f(y)] = [f(z)]$.

An instrumental part of the analysis of the map s (and consequently f) is to use a tiling (*i.e.*, branched cover) mesh M'' constructed from M' as follows. Use the transformations of the symmetry group G as defined in (3)-(4) and stitch $s(M')$ to itself along boundaries to create an infinite triangulation in the plane (with no boundary); call this triangulation M'' and the image of each vertex $v_i \in V''$ is denoted, as before, $u_i \in \mathbb{R}^2$. We

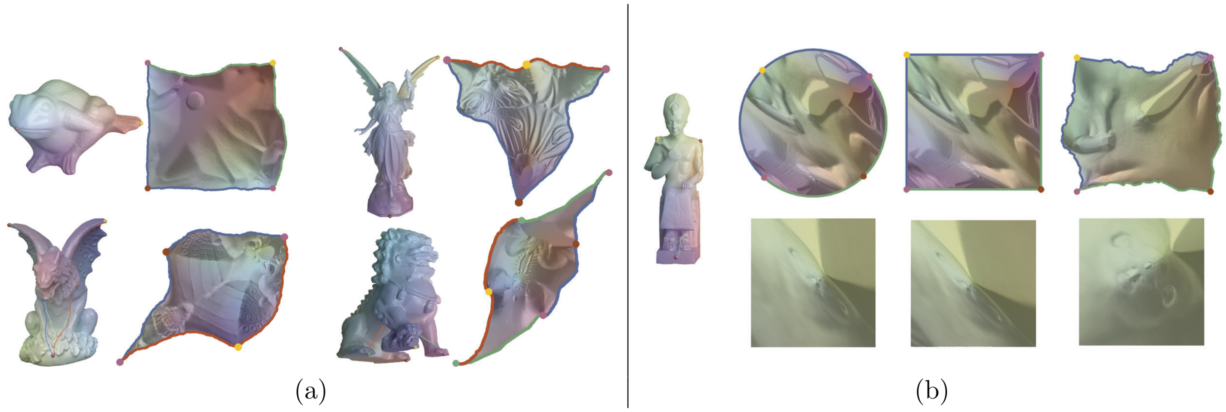


Figure 5. (a) shows convex combination homeomorphism of surface triangulations to different sphere-type orbifolds (from top-left): (244), (236), (333), and (2222). (b) shows convex homeomorphism to a convex polygonal domain (two left columns) and to a sphere-type (244) orbifold; bottom row shows blow-ups of the image. Note that the orbifold map is approximately conformal. *Images taken from* [**AL15**].

will keep denoting this simplicial map s and the mesh embedding in the plane $s(M'')$. Note that $s : M'' \to \mathbb{R}^2$ satisfies the convex combination property at *all* vertices. Indeed, it is clearly so (*i.e.*, by construction) at vertices in M'' originated from interior and boundary non-cone vertices of M'. It is also true for vertices originated from cone vertices C' of M' since at these points a point sub-group of G was applied to the neighbours of every cone point and therefore the cone points are at the centroid of their neighbours in M''. Further note that M'' is a boundaryless surface triangulation and therefore 3-connected.

Proposition 2. *The linear system $\mathcal{L}$ is non-singular.*

Proof. Assume a non-trivial solution u to the *homogenous* version of the linear system $\mathcal{L}$. Non-singularity will be proved by showing that u is necessarily trivial, $u \equiv 0$. Let $h(u) = a^T u$ be an arbitrary linear functional in $\mathbb{R}^2$.

Consider the *homogeneous* tiling $s(M'')$ of $s(M')$, *i.e.*, using the homogeneous version of (3)-(4). That is, without a translational part. Further note that u_i for all vertices originated from cone points C' are all zero. This can be understood from the homogeneous version of (2) and the homogeneous transformations (3)-(4). Lastly, note that there are only a finite number of homogeneous transformations in the symmetry groups G; in other words, the homogeneous tiling of $s(M')$ consists of infinitely repeating copies of a finite number of linear (not affine) transformations of $s(M')$.

By the comments above, the vertex function h defined by $h_i = a^T u_i$ is a convex combination function over all vertices of M''. Let $h_j = \min_i h_i$. Such a minimum exists since $s(M'')$ consists of a finite number of copies of $s(M')$. If $h_j < 0$ we can use a discrete maximum principle argument and construct a path P connecting u_j and its nearest cone vertex u_c in $s(M'')$ and conclude that $h_c = h_j < 0$, in contradiction to the homogeneous version of (2) that asserts $u_c = 0$. Repeating this argument for $h_j = \max_i h_i$ we get that $h \equiv 0$. Since a is arbitrary $u \equiv 0$.

□

Figure 5(a) shows some examples of convex combination homeomorphisms of topological spheres onto a sphere-type Euclidean orbifold; (b) shows a comparison of convex combination mappings of the same cut surface M' onto two convex polygonal domains (disk-like and square) and a sphere-type orbifold. Note how angles are better preserved in the latter mapping; this will be discussed later on. Note that in case that $\mathcal{N}$ is a convex polygonal domain only equations (a) and (c) are used and there is no need for (b). That is, all boundary vertices of M are mapped to vertices of $\Omega = \mathcal{N}$.

Theorem 1. *Let M be a 3-connected surface triangulation and $\mathcal{N} \in \mathcal{F}$ a homeomorphic target domain. Then, the simplicial map f defined by $\mathcal{L}$ is a homeomorhpism.*

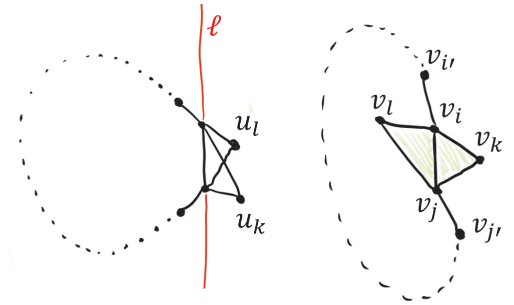


FIGURE 6. Diagram for proof of (ii) in homeomorphism proof.

3. Proof of homeomorphism.

It is not a-priori clear that $f : M \to \mathcal{N}$ constructed above is a homeomorphism. Indeed, f constructed with $\mathcal{N}$ taken to be a non-convex polygonal domain or a different Euclidean cone surface in general will not always be a homeomorphism. Next we will prove the homeomorphism property of f using arguments from different papers [**AL16, Lov04, Lip14**]. We will prove only the case of $\mathcal{N}$ being a orbifold for two reasons: first, the proof for convex polygonal domains can be easily deduced from the proof below (it is almost a particular case), and second, the proof for convex polygonal domains has many excellent versions in the literature, *e.g.*, the concise proof in [**EH10**], and the proof based on a discrete index theorem in [**GGT06**]. We also note the proof in [**Lov04**]. In the original orbifold paper [**AL15**] the proof of the embedding is done by reduction to the torus case, which is a particular instance of a Euclidean orbifold and is a sub-group of all other wallpaper groups. We provide here, hopefully, a clear, short and self-contained proof. The proof follows the following steps:

(i) The simplicial map $s : M' \to \mathbb{R}^2$ defined by the solution to $\mathcal{L}$ does not degenerate and maintains the orientation of at least one triangle of M'.
(ii) Given two triangles sharing an edge in M'. If s does not degenerate one of the triangles then it also does not degenerate the other triangle and the images of the two triangles under f will be on two different sides of the common edge.
(iii) s does not degenerate and maintains the orientations of the triangles of M' and therefore defines a homeomorphism $f : M \to \mathcal{N}$.

4. Part (i).

We will consider the mesh M'' and the tiling $s(M'')$ of $s(M')$. Consider a generic point $u \in \mathbb{R}^2$ (*i.e.*, a point not on any edge image of M'', *i.e.*, $u \notin s(E'')$). We will show it is contained in some non-degenerate positively oriented triangle $s(f_{ijk})$. To find such a triangle we can compute the winding number $\omega(u,t)$ of u w.r.t. the oriented boundary curve of some oriented triangle $t = s(f_{ijk})$. If $u \in t$ then $\omega(u,t) = \pm 1$ otherwise $\omega(u,t) = 0$. Another property of the winding number is that

$$\sum_i \omega(u, t_i) = \omega(u, \cup_i t_i). \tag{7}$$

Let us denote the diameter of $s(M')$ by $d > 0$. To find a triangle containing u let us consider a tiling of enough copies of M' around u using group transformations G (as defined in Equations (3)-(4) and Figure 2) so that: (i) any copy of M' not considered is of distance greater than d to u; and (ii) the boundary of the union of this tiling is a closed polygonal line of distance at-least d to u. From (i)+(ii) we get that the winding number of u w.r.t. the boundary of the tiling is 1. On the other hand by (7) the winding number equals the sum of windings of all triangles in the tiling. Therefore, there has to be at-least one triangle positively oriented and containing u.

5. Part (ii):

To prove (ii) we will start with the following lemma for the infinite triangulation M'':

LEMMA 1. *Let $s(M'')$ be a tiling generated from a convex combination map $s(M')$ defined as the solution of $\mathcal{L}$. Consider an infinite line $\ell \subset \mathbb{R}^2$ and vertices u_i, u_j which are strictly on one side of ℓ. Then, there is a path $P \subset E''$ connecting u_i, u_j that is also strictly on the same size of ℓ.*

Before proving this Lemma let us use it to prove (ii). Let f_{ijk}, f_{ikl} be two faces in M' sharing an edge and f_{ijk} is not degenerate. It is enough to show (ii) for a copy of these faces in M'' (since all these copies are rigid motions of the original triangles). For a bit of notational convenience we will treat $u \in \mathbb{C} \cong \mathbb{R}^2$, and $\Re(u)$ will denote the real part of u. Without loosing generality assume that $\ell = \{u \mid \Re(u) = 0\}$ and $\Re(u_k) > 0$. Assume toward contradiction that $\Re(u_l) \geq 0$, see Figure 6. By the convex combination property there are $v_{i'}, v_{j'}$ neighbors of v_i, v_j (resp.) so that $\Re(u_{i'}), \Re(u_{j'}) < 0$. Use Lemma 1 to connect $v_{i'}$ and $v_{j'}$ by a path $P = [v_{i'} = v_{p_0}, v_{p_1}, v_{p_2}, \ldots, v_{p_k}, v_{j'} = v_{p_{k+1}}]$ with $\Re(u_{p_i}) < 0$, for $i = 1, \ldots, k$. Now consider the simple closed path $Q = [v_j, v_i, P]$. The path Q divides M'' to two connected parts, one bounded and one unbounded. Consider the bounded part $U \subset V''$: it contains either v_k or v_l. Since $\Re(u_q) \leq 0$ for all $u_q \in Q$, the (second part of the) discrete maximum principle (Proposition 1) implies that $\Re(u_j) \leq 0$ for all $v_j \in U$. Since $\Re(u_k) > 0$, U cannot contain v_k.

So $v_l \in U$. In fact, v_l is in the interior of U. Indeed, $v_l \neq v_i, v_j$, and $\Re(u_l) \geq 0$ while $\Re(u_{p_i}) < 0$, $i = 0, \ldots, k+1$ so $v_l \notin P$.

There exists $U' \subset U$ a 3-connected subgraph containing v_l and a boundary point $v_b \in P$. By the 3-connectedness of M'' removing v_i, v_j keeps M'' connected and we can connect v_l to a vertex outside U using a path not passing through v_i, v_j which mean the path must intersect a vertex $v_b \in P$. Let this path be $P' = [v_l = w_0, w_1, \ldots, w_{s-1}, v_p = w_s]$. W.l.o.g. we can assume v_p is the first boundary vertex in P (since otherwise we can shorten the path). The union of the 1-neighborhoods of w_r, $r = 0, \ldots, s$ is a 3-connected graph.

Lastly, since U' is 3-connected, $\Re(u_l) \geq 0$ while there is a boundary vertex in U' so that $\Re(u_b) < 0$ we have a contradiction by the maximum principle (since $\Re(u_l) \geq 0$ attains a maximal value).

PROOF. *(Lemma 1)*
Lets assume, as before, without loosing generality that $\ell = \{u \mid \Re(u) = 0\}$. Let us first show that we can find an infinite path $P = [v_i, v_{i_1}, v_{i_2}, \ldots]$ starting from u_i so that $\Re(u_i) \leq \Re(u_{i_k}) \nearrow \infty$. Assume that v_i has a neighbor v_{i_1} so that $\Re(u_{i_1}) > \Re(u_i)$. Then from the convex combination property we can construct a strictly monotone infinite series $P = [v_i, v_{i_1}, v_{i_2}, \ldots]$. We need to show that $\Re(u_{i_k}) \to \infty$. Since M'' is made out of finite number of types of edges there is a number $\delta > 0$ so that $\Re(u_{i_{k+1}}) - \Re(u_{i_k}) \geq \delta$ and the convergence to infinity is proven. If v_i does not have such a neighbor, the convex combination property means all its neighbors are on the line $\ell + \Re(u_i)$. Since M'' is not contained in $\ell + \Re(u_i)$ there is some vertex $v_{i'}$ connected to v_i by a simple path so that $\Re(v_{i'}) > \Re(v_i)$. Now we can continue as above to construct P.
Let $P = [v_i, v_{i_1}, v_{i_2}, \ldots]$ and $Q = [v_j, v_{j_1}, v_{j_2}, \ldots]$ be two monotonic paths starting from v_i and v_j (resp.) and going to infinity. If P, Q intersect at some vertex we have a simple path connecting v_i, v_j. So lets assume they do not intersect. As before, let $d > 0$ denote the diameter of one copy of the image of M' in $\mathbb{R}^2$, $s(M')$. Therefore, any two cones v_{c_1}, v_{c_2}, or a vertex v_k and a cone v_{c_1} in the same copy of M', can be connected by a shortest path that is contained in a disk of diameter d. Therefore given two arbitrary vertices v_k, v_l, such that $\Re(u_k) \leq \Re(u_l)$ they can be connected by a path R going from v_k to a nearest cone v_{c_1}, traveling on the two dimensional grid made of cones to a nearest cone v_{c_2} to v_l and then to v_l. Note that the cone grids of all Euclidean orbifolds are regular and made out of two generators, $n\eta + m\xi$. All the vertices of R satisfy $\Re(u_r) > \Re(u_k) - 2d$. Therefore continuing P, Q until their distance from ℓ is at-least $2d$, they can be connected by a path R also to the right of ℓ. Concatenating P, R, Q^{-1}, possibly eliminating cycles will provide the desired path. □

6. Part (iii).

To prove (iii) we note that we have by now that all triangles in $s(M'')$ are non degenerate and positively oriented. We will repeat the winding number argument above to conclude that every generic point $u \in \mathbb{R}^2$ is covered by exactly one triangle $s(f_{ijk})$. Repeating the same argument we see the winding w.r.t. a sufficiently large tiling is 1. Also the triangles not participating in the tiling are too far to contain u. Since all the triangles in the tiling can contribute either 0 or 1, we conclude exactly one triangle contain u. Since s is an open map it means that every point u is covered exactly once. Therefore $s : M'' \to \mathbb{R}^2$ is a homeomorphism, and consequently $f : M \to \mathcal{N}$ is a homeomorphism on $\mathcal{N}$.

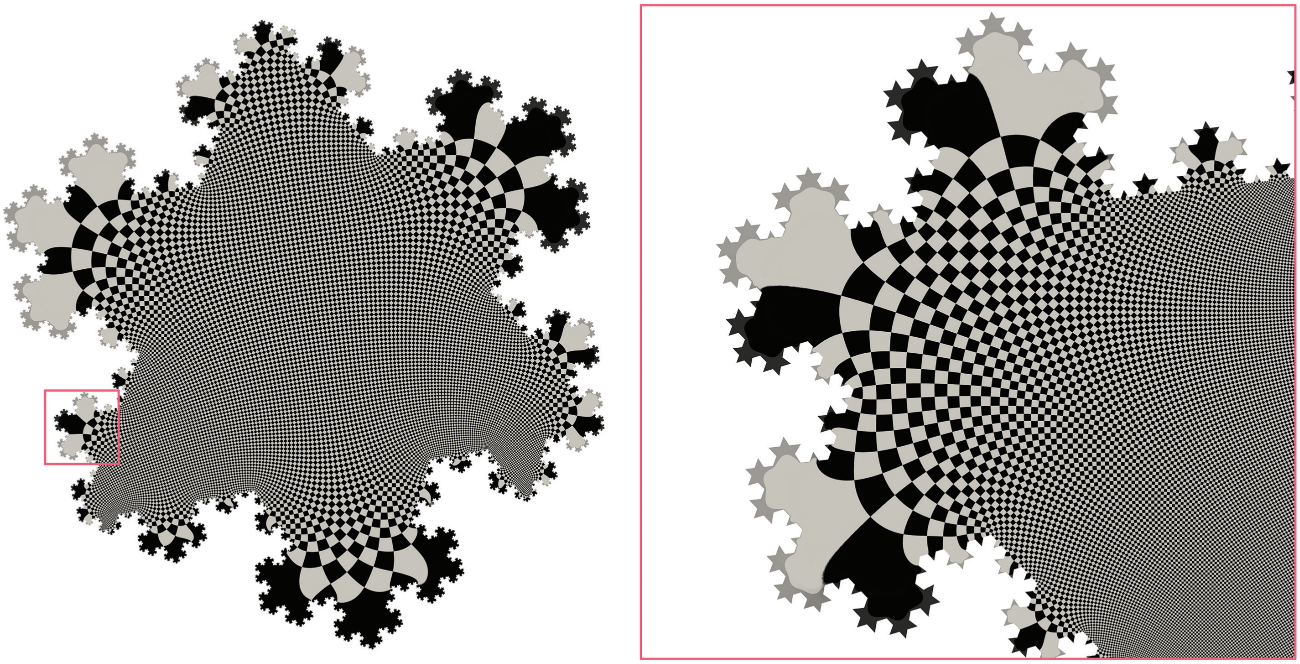


FIGURE 7. An approximation of the notorious Riemann map from a polygonal approximation of the Koch snow-flake (computed with 6 recursions) to a triangle. The approximation is computed with a 6 million vertex mesh and captures different resolutions of this map as shown in the blow-up on the right. *Image taken from* [**DSL19**].

7. Variational principle.

When the weights are symmetric, $w_{ij} = w_{ji}$, convex combination maps have variational formulation. Denote the discrete Dirichlet energy by

$$E_D(u) = \frac{1}{2} \sum_{e_{ij} \in E} w_{ij} (u_i - u_j)^2. \tag{8}$$

The linear system of equations $\mathcal{L}$ characterizes the solution of the following variational problem:

$$\begin{aligned} &\min_u && E_D(u) && && \text{(9a)} \\ &\text{s.t.} && a_i^T u_i + b_i = 0, && \forall \langle v_i \rangle \in V'_{BS} && \text{(9b)} \\ & && u_i = r_{ii'} u_{i'} + t_{ii'}, && \forall \langle v_i \rangle \in V'_{BP} && \text{(9c)} \\ & && u_i = p_i, && v_i \in C' && \text{(9d)} \end{aligned}$$

One way [**PP93, Flo03a**] to figure out a good choice of weights w_{ij} is to choose it so that (8) becomes the Dirichlet energy, $\int_M |\nabla f|^2$, when computed on a piecewise linear simplicial map f. A calculation shows that in this case

$$w_{ij} = \cot \alpha_{ij} + \cot \beta_{ij},$$

where α_{ij}, β_{ij} are the angles opposite to the edge e_{ij} in M. These weights are called the cotan weights for obvious reasons. The cotan weights are symmetric and since $w_{ij} = \frac{\sin(\alpha_{ij}+\beta_{ij})}{\sin \alpha_{ij} \sin \beta_{ij}}$ they are positive iff the sum of angles supporting e_{ij} is smaller than π; such a triangulation M is called *Delaunay* (without 4 cocircle points). Using non-positive weights will generically damage the homeomorphism property of the map f but approximation properties will not be affected (discussed later). Another interesting option of weights which are not symmetric but always positive are the mean-value weights [**Flo03b**]. Although they don't possess a variational principle due to the non-symmetry these weights can be used in the linear system $\mathcal{L}$ to produce homeomorphic convex combination mappings.

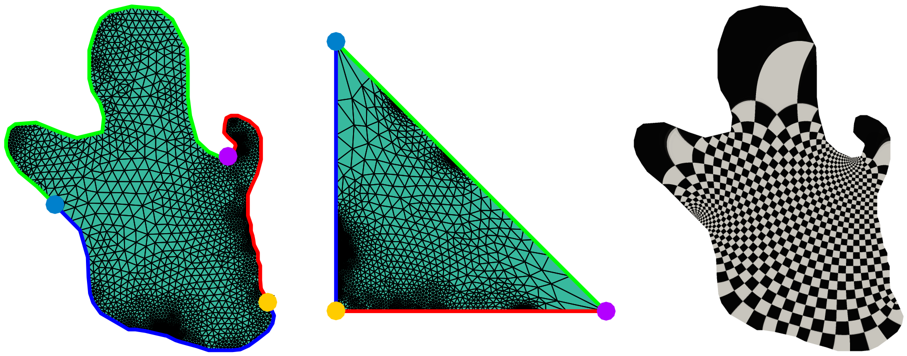


FIGURE 8. Convex combination homeomorphism, mapping a triangulation of a simply connected polygonal domain (left) onto a Euclidean orbifold triangle (middle). On the right, a pull-back checkerboard texture via the inverse of the mapping to visualize the conformality of the discrete mapping.

8. Approximation of conformal mappings.

The variational representation (9) leads to an interesting observation in case w_{ij} are the cotan weights [**AL15**]. The Dirichlet energy can be seen as an upper-bound to the area functional,

$$E_D(u) \geq E_A(u),$$

where E_A denotes the area functional, measuring the sum of (positive) triangle areas in $s(M')$. In case M is 3-connected and Delaunay, $w_{ij} > 0$, and Theorem 1 implies that $E_A(u) = \text{area}(\Omega) = \text{area}(\mathcal{N})$. The difference

$$E_C(u) = E_D(u) - E_A(u)$$

can be seen as the conformal distortion, and in the case u is a homeomorphism it equals the so-called least-squares conformal energy [**LPRM02**]. When using $\mathcal{N} \in \mathcal{F}$ with three cone points the number of constrained points exactly matches the degrees of freedom specified by the Riemann mapping theorem. Therefore, one can ask if $f : M \to \mathcal{N}$ actually converges under refinement to the unique conformal map, mapping three prescribed points in M to the cones of $\mathcal{N}$. This was proved recently in [**DSL19**] for the case of a simply connected polygonal domain in the plane, and when $\mathcal{N} \in \mathcal{F}$ is a triangle. In fact it is shown that convergence in H^1 holds for any triangle (not just ones that can tile the plane) and any mesh (not necessarily Delaunay) however it is uniform when M is Delaunay and $\mathcal{N} \in \mathcal{F}$ (so f is a homeomorphism):

THEOREM 2. *Let $\mathcal{P} \subset \mathbb{R}^2$ be a simply-connected polygonal domain and $\mathcal{N}$ a triangle. Let M_h be surface triangulations of $\mathcal{P}$ with maximal edge length $h \to 0$ and all angles of the triangulations bounded below by some $\delta > 0$. Let $f_h : \mathcal{P} \to \mathcal{N}$ denote the simplicial maps defined by solving $\mathcal{L}$ fixing three pre-images in $\partial\mathcal{P}$ to the cones of the triangle. Let $\Phi : \mathcal{P} \to \mathcal{N}$ be the Riemann map fixing the same pre-images. Then,*

$$\|f_h - \Phi\|_{H^1} \to 0.$$

Furthermore, if M_h are Delaunay and 3-connected, then the convergence is also uniform.

Figures 7,8 show examples of approximations of Riemann mappings from polygonal domains to Euclidean orbifolds; the checkerboard texture visualizes the conformality of the simplicial mappings.

9. Surface to surface mappings.

Another application of convex combination mappings is Problem 1, namely construction of a homeomorphism $f : M_1 \to M_2$ between two surface triangulations M_1, M_2. Consider a cone surface $\mathcal{N} \in \mathcal{F}$ that is topologically equivalent to both M_1 and M_2. Construct simplicial homeomorphisms $f_1 : M_1 \to \mathcal{N}$ and $f_2 : M_2 \to \mathcal{N}$ and consider $f = f_2^{-1} \circ f_1 : M_1 \to M_2$. The map f can be seen as simplicial if one considers an isomorphic joint subdivision (a.k.a. meta-mesh) M of M_1 and M_2. Using $\mathcal{N}$ with three cones provides a simplicial homeomorphism approximating the conformal map interpolating arbitrary three vertices between M_1 and M_2. (Note that M_1, M_2 are in particular Riemann surfaces.) Using $\mathcal{N} \in \mathcal{F}$ with four cones (*i.e.*, orbifold of type (2222)) provides a quasiconformal map $f : M_1 \to M_2$ with approximately constant conformal distortion. Therefore, the orbifolds $\mathcal{N}$ with the four cone points provide a quasiconformal approximation to the Teichmüller map interpolating four points on the sphere [**AEK$^+$66**]. Indeed, it is possible to write $f = f_2^{-1} \circ A \circ f_1$, where f_1, f_2 are homeomophisms

FIGURE 9. A simplicial map between pairs of surface triangulations composed out of two convex combination mappings to a common orbifold (here (2222)); the colored spheres indicate the interpolated points $(x_i, y_i) \in M_1 \times M_2$ in the map. *Images taken from* [**AL15**].

of finite surface triangulations, which makes them quasiconformal, and since f_1, f_2 approximate conformal mappings their conformal distortion is close to 0; A is an affine map. Figure 9 depicts examples of approximate quasiconformal mapping between pairs of surface triangulations interpolating four landmark points.

10. Other Euclidean cone surfaces.

PROBLEM 2. *When are convex combination mappings onto Euclidean cone surfaces not in $\mathcal{F}$ homeomorphisms?*

This could be useful for discrete mappings for two reasons: (i) it will allow constraining more than three or four points; and (ii) it will potentially allow choosing $\mathcal{N}$ so to reduce isometric distortion exerted by the map. As far as the author knows there are no Euclidean cone surfaces outside the family $\mathcal{F}$ that generically allow homeomorphic convex combination maps. Furthermore, an interesting (and not completely solved) problem is:

PROBLEM 3. *Find a characterization of when a Euclidean cone surface provides a homeomorphic convex combination map, even if it does not allow it generically.*

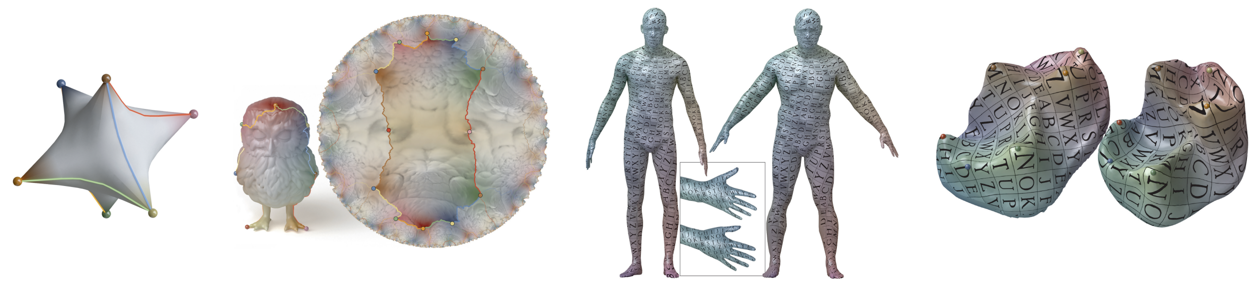
Let us review several known results addressing this latter problem discussing other Euclidean cone surfaces $\mathcal{N} \notin \mathcal{F}$. In [**GGT06**] non-convex polygonal domains are discussed and a sufficient condition for a convex combination map to be bijective in this case is that every reflex cone (*i.e.*, a cone x with angle sum $\theta(x) > \pi$) is in the convex hull of its neighbors. (Notice that this latter condition can fail.) A similar generalization to multiply-connected domains also exists.

In [**BCW17**] a Euclidean cone manifold with rational cone angles $k\frac{2\pi}{q}$, $k, q \in \mathbb{N}$ is considered and it is shown that if the triangles adjacent to the cones are positively oriented then the convex combination map is locally a homeomorphism.

In [**TFV$^+$13, AL16**] convex combination mappings are considered into the hyperbolic plane by minimizing the discrete hyperbolic Dirichlet energy [**SZS$^+$13**]:

$$E_D(u) = \frac{1}{2} \sum_{e_{ij} \in E} w_{ij}\, d(u_i, u_j)^2,$$

where $u_i \in \mathcal{H}$ are points in the hyperbolic plane and $d(\cdot, \cdot)$ denotes the hyperbolic distance. Boundary conditions can be added to form a variational problem generalizing (9) to the hyperbolic case, namely for defining and computing homeomorphic simplicial mappings of a surface triangulation onto one of the hyperbolic orbifolds $\{\mathcal{N}\}$. Hyperbolic orbifolds, are defined similarly to the Euclidean orbifolds as $\mathcal{H}/G$ where G is a symmetry group of the hyperbolic plane. In contrast to Euclidean orbifolds, hyperbolic orbifolds are an infinite family of (hyperbolic) cone surfaces that include arbitrary genus surfaces and arbitrary number of cones. The boundary conditions are similar to (9b)-(9d) forcing identified boundary vertices of M' to correspond via the relevant hyperbolic isometries, namely Möbius transformations. It is proved in [**AL16**], basically following the same proof as the Euclidean case above (this time in the Klein hyperbolic model), that critical points of this non-linear problem provide homeomorphisms onto the respective hyperbolic orbifold. Figure 10(a) shows a sphere-type hyperbolic orbifold with seven cones of angle π, *i.e.*, with symbol (2^7); (b) shows a homeomorphic hyperbolic convex combination mapping to this hyperbolic orbifold; (c) shows a homeomorphism $f = f_2^{-1} \circ f_1 : M_1 \to M_2$



(a) (b) (c) (d)

Figure 10. (a) shows (an approximation) of a sphere-type hyperbolic orbifold with seven cones of angle π; (b) hyperbolic convex combination mapping of a sphere-type surface triangulation (owl) to this target orbifold; (c)-(d) homeomorphisms of surface triangulations by composing hyperbolic convex combination mappings to a common orbifold. *Images taken from* [**AL16, MGA+17**].

between two human surface triangulations M_1, M_2 interpolating seven landmark points, where $f_i : M_i \to \mathcal{N}$, $i = 1, 2$ are hyperbolic convex combination mapping to the same hyperbolic orbifolds. Note that allowing the interpolation of seven points produces a more faithful map than the one generated with four interpolated points in the Euclidean case, Figure 9, right. Figure 10(d) shows another application of mapping anatomical surfaces (teeth).

11. Convex combination in higher dimensions.

A natural question is the generalization of convex combination mapping to three dimensional simplicial complexes. Consider a tetrahedral mesh $M = (V, E, F, T)$, where $T = \{t_{ijkl}\}$ is the tetrahedra set, $t_{ijkl} = \text{hull}\{v_i, v_j, v_k, v_l\}$. Convex combination mappings can be defined as before using (1). Unfortunately, even in the most basic case of a convex polyhedron boundary conditions (assuming M is topologically a ball) the convex combination map is not guaranteed to be a homeomorphism, and counter examples were found [**DVPV03, FPT06**]. x

12. Acknowledgements.

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