Proceedings of Symposia in Applied Mathematics

Discrete Parametric Surfaces

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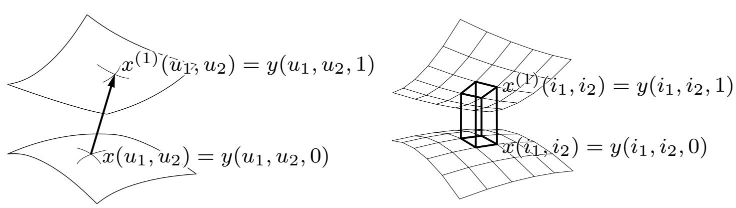
1. Introduction

Discrete parametric surfaces are discrete analogues of smooth parametric surfaces. They are, however, not simply discrete approximations of their smooth counterparts, but are the subject of a separate discrete theory. As it turns out, a systematic theory of parametric surfaces can be based on integrable systems, and the discrete case can be interpreted as a "master" case which contains smooth surfaces as a limit.

This section on discrete parametric surfaces is organized as follows: We first introduce notation. Two particular kinds of discrete surfaces are discussed next: circular nets in §2, and K-nets in §3 are examples of a 3-system and a 2-system, respectively. In the case of K-nets, we also discuss the relation to the sine-Gordon equation. We then show applications within mathematics in §4, cf. [**BHS06**], and the connection with freeform architecture in §5, cf. [**PW16**]. The main source for this chapter is the monograph [**BS09**].

1.1. Discrete, semidiscrete, and continuous surfaces. The simplest case of a continuous surface is a point $x(u_1, u_2)$ of space depending on two parameters u_1, u_2. A discrete surface $x(i_1, i_2)$ is the same, only the parameters i_1, i_2 are integers. The notion of *transformation* of a surface usually refers to a pair $x(u_1, u_2)$ and $x^{(1)}(u_1, u_2)$ of surfaces which are in a certain relation (images taken from [**BS09**]):

(1)


The theory of transformations was fully developed in the 1920s, cf. [**Eis23**]. Well known instances are Darboux transforms, Bäcklund transforms, or the Christoffel transform (which relates minimal surfaces with conformal mappings into sphere, and leads to the Weierstrass representation of minimal surfaces in terms of meromorphic functions). A sequence of surfaces created by iterated application of a

1991 *Mathematics Subject Classification.* Primary 14J25; Secondary 37J35.
Key words and phrases. Discrete differential geometry, integrable systems.

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transformation rule constitutes a *semidiscrete* object $y(u_1, u_2, i_3)$, where i_3 is an integer parameter:

$$y(u_1, u_2, 0) = x(u_1, u_2), \quad y(u_1, u_2, 1) = x^{(1)}(u_1, u_2), \quad y(u_1, u_2, 2) = x^{(11)}(u_1, u_2), \ \ldots$$

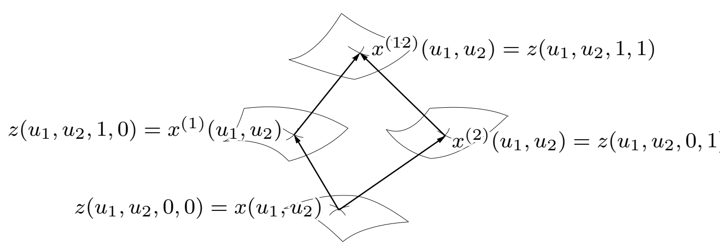
Transformations of discrete surfaces can be treated in a similar way: With

$$y(i_1, i_2, 0) = x(i_1, i_2), \quad y(i_1, i_2, 1) = x^{(1)}(i_1, i_2), \quad y(i_1, i_2, 2) = x^{(11)}(i_1, i_2), \ \ldots$$

we define a mapping y from $\mathbb{Z}^3$ to space. We see that a sequence of discrete k-dimensional surfaces is nothing but a discrete $(k+1)$-dimensional surface, and the special role of the first two parameters disappears.

A particular feature of many transformations is that they enjoy *Bianchi permutability*: If x has transforms $x^{(1)}$ and $x^{(2)}$, then there exists $x^{(12)}$ which is a transform of both $x^{(1)}$ and $x^{(2)}$ simultaneously (image taken from [**BS09**]):

(2)



By iterating this procedure we create a two-dimensional lattice of surfaces, i.e., a mapping "z" from $\mathbb{R}^2 \times \mathbb{Z}^2$ to space. It can be seen as a semidiscrete 4-dimensional surface $z(u_1, u_2, i_3, i_4)$. It is very interesting that the 2D discrete surfaces $x(i_1, i_2) = z(u_1, u_2, i_3, i_4)$ contained in this 4-dimensional semidiscrete object typically exhibit geometric properties similar to discrete surfaces: *The transformations which apply to a class of smooth surfaces provide guidelines on how to find a class of analogous discrete surfaces.*

However if x is a discrete surface to begin with, a 2-dimensional lattice of surfaces is nothing but a 4-dimensional discrete surface $z(i_1, i_2, i_3, i_4)$. The special role of the first two parameters disappears.

It is a key principle of discrete differential geometry that the smooth theory can be obtained by a limit process from the discrete one. We can let a 2D discrete surface converge to a smooth surface, a 3D discrete surface to a sequence of smooth surfaces, and so on.

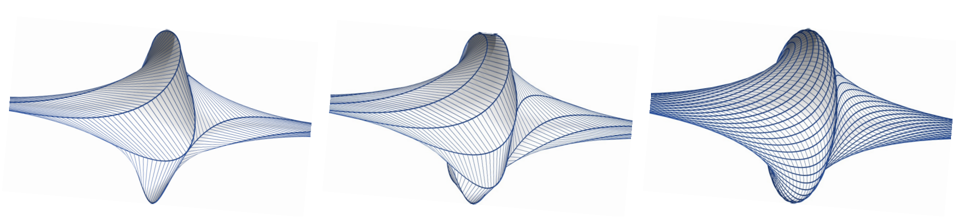


FIGURE 1. Discrete K-surfaces. Their defining property is that in each face, opposite edges have equal length, and in each vertex, all four incident edges are co-planar. Discrete surfaces can converge to semidiscrete or continuous surfaces.

1.2. History. Integrable equations. An important feature of parametric surfaces is the relation to integrable equations. This connection has a long tradition. The best known example concerns a special "asymptotic" parametrization of surfaces whose Gauss curvature equals the constant -1. It turns out that the angle $\phi(u_1, u_2)$ between parameter lines obeys the sine-Gordon equation

$$\partial_{12}\phi = \sin\phi. \tag{3}$$

It is a fact that for any solution $\phi(u_1, u_2)$ of this equation, another solution $\phi^{(1)}$, called the *Bäcklund transform* of ϕ, can be defined by

$$\partial_1\phi^{(1)} = \partial_1\phi + 2a\sin\frac{\phi+\phi^{(1)}}{2},\ \partial_2\phi^{(1)} = -\partial_2\phi + \frac{2}{a}\sin\frac{\phi^{(1)}-\phi}{2}.$$

This Bäcklund transformation of functions corresponds directly to the Bäcklund transformation of surfaces: ϕ and $\phi^{(1)}$ are angle functions associated with a Bäcklund pair of surfaces x and $x^{(1)}$. There is even a Bianchi permutability theorem analogous to the surface case: With the arrow symbolizing the Bäcklund relation, we have

$$\begin{tikzcd}[row sep=small] & \phi^{(1)} \\ \phi \arrow[ur] \arrow[dr] & \\ & \phi^{(2)} \end{tikzcd} \implies \exists\, \phi^{(12)} \text{ such that } \begin{tikzcd}[row sep=small] & \phi^{(1)} \arrow[dr] & \\ \phi \arrow[ur] \arrow[dr] & & \phi^{(12)}. \\ & \phi^{(2)} \arrow[ur] & \end{tikzcd} \tag{4}$$

Surfaces of constant Gaussian curvature have been instrumental in the development of the systematic "integrability" theory of discrete surfaces. The original Bäcklund transform of smooth K-surfaces was published in 1883 [**Bäck83**], and the permutablity theorem soon after [**Bia02**]. Discrete K-surfaces were constructed in the 1950's [**Sau50, Wun51**]. The connection of discrete surfaces to the discrete Hirota equation, which is a discrete analogue of the sine-Gordon equation, was revealed by [**BP96**]. We are coming back to this topic in §3.

Typically, for any particular class of surfaces (e.g. the circular nets of §2 or the K-nets of §3) one needs some elementary geometric construction in order to establish multidimensional discrete surfaces. Since the discrete case serves as a master theory from which the continuous one is obtained by a limit process, that elementary geometric theorem can be seen as the discrete analogue of the various kinds of integrability needed to construct surfaces and their transformations.

1.3. Curvatures. Surface classes can be equivalently characterized by different properties. E.g., a surface is minimal $\iff$ it is locally area-minimizing $\iff$ its mean curvature vanishes $\iff$ it has a parametrization in terms of of meromorphic functions $f(z)$ and $g(z)$ such that fg^2 is holomorphic, with $x(u,v) = \frac{1}{2}\operatorname{Re}\int_0^{u+iv}\big(f(z)(1-g(z)^2),\, if(z)(1+g(z)^2),\, 2f(z)g(z)\big)\,dz$.

It is this Weierstrass representation which leads to a discretization in the context of *parametric* surfaces (see §4). Area minimization is the basis of a different discretization, which is not parametric [**PP93**].

It is a curious fact that many classes of surfaces which are originally defined via *curvature* (K-surfaces, minimal surfaces, cmc surfaces) had well-established discrete counterparts which were defined by an equivalent characterization not involving curvatures. However, meanwhile curvatures for discrete surfaces have been studied

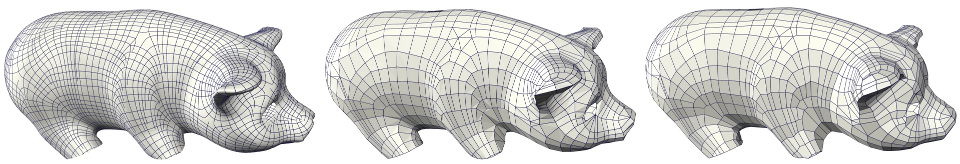


FIGURE 2. *Left:* A conjugate parametric surface $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ covers most of this geometric shape. *Center:* discrete parametric surfaces extracted by sampling have elementary quadrilaterals which are almost planar. *Right:* Global optimization makes all faces planar. This optimization problem is numerically feasible because we are already very close to a solution.

in a systematic way, and there are general concepts of curvatures which apply to the discrete surface classes mentioned above [**BPW10, HSFW17**].

1.4. Discretization principles. The difficulty of assigning curvatures illustrates an important issue: It is not clear a priori which of the various equivalent properties of a class of smooth surfaces should be the one which guides the discretization. However, a discretization is good, or worthy of further investigation, if not just one property carries over from the smooth to the discrete setting, but more than one. For example, the variational definition of discrete minimal surfaces by [**PP93**] leads to simplicial minimal surfaces which not only minimize area, but which also exhibit an associated family of minimal surfaces. The guiding principle of discretization in the case of *parametric* surfaces often is integrability, or (in the discrete case), so-called multidimensional consistency.

1.5. Notation for continuous and discrete nets. A smooth parametric surface maps a parameter value $u \in \mathbb{R}^d$ to a point $x(u) \in \mathbb{R}^n$. Derivatives are described by the symbols

$$\partial_k x,\ \partial_{kl} x, \ldots$$

A discrete parametric surface maps an *integer* parameter value $u = (u_1, \ldots, u_d) \in \mathbb{Z}^d$ to a point $x(u) \in \mathbb{R}^n$. The role of derivatives is played by differences, e.g.

$$\Delta_1 x(u_1, u_2, \ldots, u_d) = x(u_1 + 1, u_2, \ldots, u_d) - x(u_1, u_2, \ldots, u_d),$$
$$\Delta_2 x(u_1, u_2, \ldots, u_d) = x(u_1, u_2 + 1, \ldots, u_d) - x(u_1, u_2, \ldots, u_d).$$

In general, we use a lower index i to indicate a forward shift in the i-th parameter, and the index $\bar{i}$ to indicate a backward shift.

$$x_k(\ldots, x_k, \ldots) = x(\ldots, u_k + 1, \ldots) \qquad x_{\bar{k}}(\ldots, x_k, \ldots) = x(\ldots, u_k - 1, \ldots)$$

We use the notation x_{kk}, x_{kl}, $x_{\bar{k}l}$, and so on for iterated shifts. E.g. in a 2-dimensional net $x(u_1, u_2)$ we use a diagram like the one shown in the inset figure to indicate the immediate neighborhood of a general point $x(u_1, u_2)$. The forward difference operator is expressed as $\Delta_k x = x_k - x$.

$$\begin{array}{ccccc} x_{\bar{1}2} & - & x_2 & - & x_{12} \\ | & & | & & | \\ x_{\bar{1}} & - & x & - & x_1 \\ | & & | & & | \\ x_{\bar{1}\bar{2}} & - & x_{\bar{2}} & - & x_{1\bar{2}} \end{array}$$

1.6. Examples. Conjugate surfaces. A *conjugate* parametrization is one where infinitesimal parameter quads are planar. For 2-surfaces, this means that we require

$$\begin{aligned}
&3\,\mathrm{vol}\left(\text{c.h.}\left(x(u),\ x(u_1+\varepsilon,u_2),\ x(u_1,u_2+\varepsilon),\ x(u_1+\varepsilon,u_2+\varepsilon)\right)\right)/\varepsilon^4\\
&= \det(x(u_1+\varepsilon,u_2)-x(u),\ x(u_1,u_2+\varepsilon)-x(u),\ x(u_1+\varepsilon,u_2+\varepsilon)-x(u))/2\varepsilon^4\\
&\approx \det(\varepsilon\partial_1 x,\ \varepsilon\partial_2 x,\ \varepsilon\partial_1 x+\varepsilon\partial_2 x+2\varepsilon^2\partial_{12}x)/2\varepsilon^4\\
&= \det(\partial_1 x,\partial_2,\partial_{12}x)=0
\end{aligned}$$

The "$\approx$" sign means equality up to remainder terms in the Taylor expansion as $\varepsilon \to 0$. We say that all infinitesimal elementary quadrilaterals of the net are planar. It is obvious how to translate the defining property

$$\det(\partial_k x, \partial_l x, \partial_{kl} x) = 0$$

to the discrete case: a discrete conjugate net is defined by

$$\det(\Delta_k x, \Delta_l x, \Delta_{kl} x) = 0.$$

Generically this is equivalent to

$$\begin{matrix} x_l & \text{—} & x_{kl} \\ | & & | \\ x & \text{—} & x_k \end{matrix} \qquad \text{co-planar,} \tag{5}$$

or to existence of coefficients c^{lk}, c^{kl} associated with the elementary quadrilateral $xx_kx_lx_{kl}$, such that

$$\Delta_k\Delta_l x = c^{lk}\Delta_k x + c^{kl}\Delta_l x.$$

For reasons which become apparent later, we use this property as definition of a discrete conjugate surface (a synonym is *conjugate net*).

2. Circular nets: a 3-system

There are two major classes of discrete parametric surfaces guided by the discretization principle of multidimensional consistency. These are the 2-systems and the 3-systems, and we explain these notions by means of prominent examples: the K-nets for the 2-systems, and the circular nets for the 3-systems. We begin with the 3-system case.

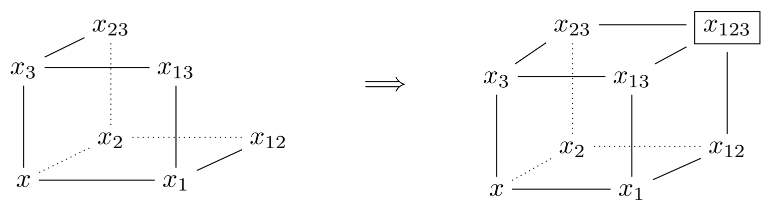
Definition 2.1. *A circular net is a discrete parametric surface where all elementary quadrilaterals are co-circular, i.e., for every vertex x and neighbors x_k, x_l, there is a circle which contains the four vertices x, x_k, x_l, x_{kl}.*

Obviously, a circular net is also conjugate. The following result concerning existence of conjugate nets and circular nets implicitly contains the definition of a 3-system:

Theorem 2.2. *Conjugacy of nets is a 3-system: Generically, a conjugate net $x(u_1, u_2, u_3)$ is uniquely determined by arbitrary initial values $x(0, u_2, u_3)$ and $x(u_1, 0, u_3)$ and $x(u_1, u_2, 0)$. The same applies to circularity.*

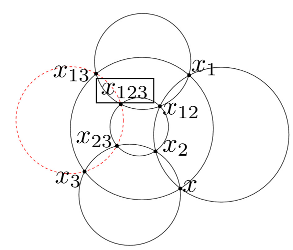
Figure 3. *Left:* 2D circular net. *Right:* An elementary cell of a 3D circular net.

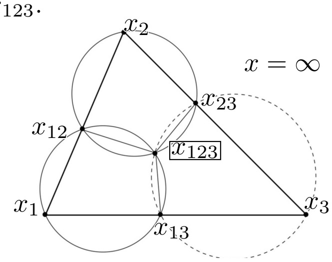
Proof. Consider a combinatorial cube with diagonal x—x_{123}, see the figure below. Assume that the quadrilaterals $xx_1x_2x_{12}$, $xx_1x_3x_{13}$, $xx_2x_3x_{23}$ adjacent to x are planar. Does there exist x_{123} such that the three quads adjacent to x_{123} are planar?



The answer is trivially yes for a conjugate net in $\mathbb{R}P^n$, $n \geq 3$, since the planes carrying the three missing faces are given by three vertices each, and x_{123} is the intersection of these planes.

For circular nets, this is not so obvious: The circumcircles of the three quads incident to x_{123} are given by three points each, but do they intersect in a common point x_{123}? The answer is yes: whenever the three quads $xx_kx_lx_{kl}$ have a circumcircle, then the three circumcircles of x_1, x_{12}, x_{13}, x_2, x_{23}, x_{21}, x_3, x_{31}, x_{32} meet in a common point x_{123}.





For a proof we observe that the statement is not affected by Möbius transformations. We therefore apply an inversion to move x to infinity. Circles passing through x become straight lines, so x_{kl} becomes a point of the edge x_kx_l of the triangle $x_1x_2x_3$. The statement about circumcircles is now shown by using elementary geometry. $\square$

The proof makes it clear that the "right" geometric setting for conjugate nets is projective space, whereas circular nets should be treated in Möbius geometry.

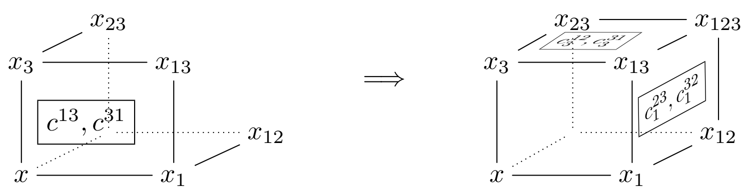
Proposition 2.3. *Assume that $x(i_1, i_2, i_3)$ is a conjugate net, and that all elementary quadrilaterals which contain vertices with $i_1 = 0$ or $i_2 = 0$ or $i_3 = 0$ are* *circular. Then, generically, circularity propagates through the net: all elementary quads are circular. The same is true for n-dimensional nets with $n \geq 3$.*

PROOF. It is sufficient to show this statement for a 3-cell with diagonal x—x_{123} where the three quads incident to x are circular. If we require circularity, then x_{123} is uniquely determined as the intersection of circumcircles of the quads incident to it. But x_{123} is *already* uniquely determined by the intersection of the planes which carry those quads. It follows that x_{123} is the same regardless if it is computed via the 3-system "conjugacy" or via the 3-system "circularity". □

LEMMA 2.4. *Consider the coefficient functions used in the definition of a conjugate net, see* (5), $\Delta_k\Delta_l x = c^{lk}\Delta_k x + c^{kl}\Delta_l x$. *For a conjugate 3-net, there is a birational mapping*

$$(c^{12}, c^{21}, c^{23}, c^{32}, c^{31}, c^{13}) \overset{\phi}{\longmapsto} (c_3^{12}, c_3^{21}, c_1^{23}, c_1^{32}, c_2^{31}, c_2^{13})$$

of the coefficients in quads incident to x, to coefficients in quads incident to x_{123}.



PROOF. We use the "product rule" $\Delta_j(a \cdot b) = a_j \cdot \Delta_j b + \Delta_j a \cdot b$ to expand

$$\begin{aligned}\Delta_i\Delta_j\Delta_k x &= \Delta_i(c^{kj}\Delta_j x + c^{jk}\Delta_k x) = c_i^{kj}\Delta_i\Delta_j x + \Delta_i c^{kj}\Delta_j x + \cdots \\ &= (c_i^{kj}c^{ji} + c_i^{jk}c^{ki})\Delta_i x + (c_i^{kj}c^{ij} + \Delta_i c^{kj})\Delta_j x + (c_i^{jk}c^{ik} + \Delta_i c^{jk})\Delta_k x.\end{aligned}$$

Permuting indices yields

$$\begin{aligned}\Delta_j\Delta_k\Delta_i x &= (c_j^{ik}c^{kj} + c_j^{ki}c^{ij})\Delta_j x + (c_j^{ik}c^{jk} + \Delta_j c^{ik})\Delta_k x + (c_j^{ki}c^{ji} + \Delta_j c^{ki})\Delta_i x. \\ \Delta_k\Delta_i\Delta_j x &= (c_k^{ji}c^{ik} + c_k^{ij}c^{jk})\Delta_k x + (c_k^{ji}c^{ki} + \Delta_k c^{ji})\Delta_i x + (c_k^{ij}c^{kj} + \Delta_k c^{ij})\Delta_j x.\end{aligned}$$

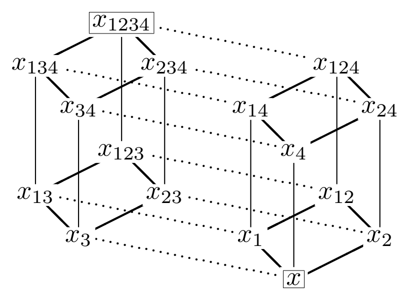
Since $\Delta_i\Delta_j\Delta_k$ is invariant w.r.t. permutations of indices, and generically the first derivatives are linearly independent, we get

$$\Delta_i c^{jk} = c_k^{ij}c^{jk} + c_k^{ji}c^{ik} - c_i^{jk}c^{ik} \quad (i \neq j \neq k \neq i).$$

These are 6 linear equations for the 6 variables $(c_3^{12}, \ldots, c_1^{32})$. Their explicit solution by matrix inversion yields the desired rational mapping ϕ, and analogously for the inverse ϕ^{-1}. Even if ϕ was derived under the assumption of linearly independent first derivatives, it may be applied to all input data. □

The following result implicitly defines an *integrable* discrete 3-system:

THEOREM 2.5. *Conjugacy is a 4-consistent (i.e., integrable) 3-system: If in the combinatorial 4-cube with diagonal $x \ldots x_{1234}$, conjugacy is imposed on the faces incident to x (and by the 3-system property , on the four 3-cubes incident to x), then there is a generically unique choice of x_{1234} such that also the remaining quads of the hypercube enjoy the property. Analogously, circularity is 4-consistent.*



Proof. The hypercube with diagonal x—x_{1234} contains four 3-cubes incident with x_{1234}. Within each of these cubes, x_{1234} is found as the intersection of 3 quads. E.g. from the cube with diagonal x—x_{123} we get

$$x_{1234} \in \operatorname{span}(x_{12}x_{123}x_{124}) \cap \operatorname{span}(x_{13}x_{132}x_{134}) \cap \operatorname{span}(x_{14}x_{142}x_{143}). \tag{6}$$

The cubes with diagonals x—x_{124}, x—x_{134}, and x—x_{224} yield three more ways to express x_{1234} as intersection of three planes. We must show that all these are equal.

(1) Each of the quads mentioned above is the intersection of two adjacent 3-cubes. Counting shows that these six cubes are actually all four 3-cubes incident with x_{1234}.

(2) Thus, in case of dimension ≥ 4 and general position, (6) is transformed into an intersection of four 3-spaces (each is the span of a 3-cube). Since all 3-cubes incident to x_{1234} occur in this expression, the transformed expression is invariant w.r.t. permutation of indices $\implies$ all ways of computing x_{1234} yield the same result.

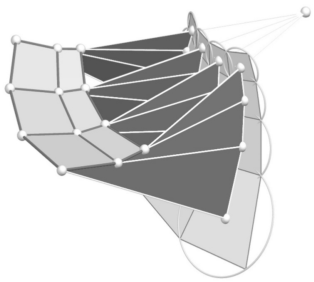
(3) x_{1234} can also be found in an alternative way, namely by using the coefficients c^{ij} defined by (5). We can compute them in the quads incident to x, apply ϕ, and compute vertices $x_{123}, x_{124}, x_{134}, x_{234}$. Repetition of this procedure for the quads incident with x_{1234} yields four different expressions for x_{1234}, within each of the four cubes incident to x_{1234}, i.e., there are four rational functions of the arguments x, x_i, x_{ij} which in the general position case and dimension ≥ 4 evaluate to the same point x_{1234}, by (2).

(4) Since almost all arguments have the general position property, the rational functions constructed above are equal. They are independent of the dimension, so they can be applied also in 3-space, proving 4-consistency.

Consistency of *circularity* follows as a corollary, since circularity propagates by Prop. 2.3. □

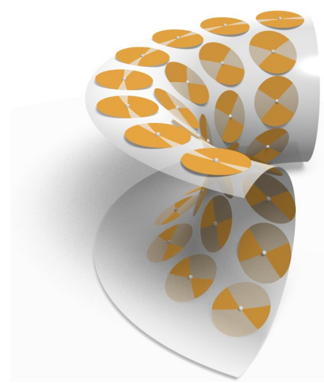
An easy combinatorial argument even shows that 4-consistency of 3-systems implies n-consistency for all $n \geq 4$.

2.1. Principal curvature lines. We have not yet mentioned the smooth class of surfaces which corresponds to the circular nets: It is the parametrization of surfaces along principal curvature lines. There are several reasons for that: Circularity of quads is a discrete version of *orthogonality*, but a more compelling reason is the behavior of the surface normals. A curve $c(t)$ on a surface is a principal curvature line, if progress in that direction causes the unit normal vector to change in the same direction ($\frac{d}{dt}n(c(t)) = \lambda(t)\frac{d}{dt}c(t)$, the tangent vectors of these curves are eigenvectors of the Weingarten map $-dn$), so the surface traced out by the surface normals is developable.



Developability means that a surface normal $N(c(t))$ and its infinitesimal neighbor $N(c(t+dt))$ “intersect” each other (in the same way an infinitesimal quad of a conjugate net is planar). It is this surface of normals which has a proper discrete analogue: As we progress along a sequence of adjacent faces, the successive axes of circles intersect, which yields a discrete developable surface. Further, convergence of circular nets to principal parametrizations can be shown [**BS09**].

3. K-nets: a 2-system



3.1. Asymptotic nets. Asymptotic parametrizations have elementary quads which are as non-planar as possible. For a negatively curved 2-surface in $\mathbb{R}^3$, the asymptotic tangents in a point are found by intersecting the surface with its own tangent plane; the parameter lines of the asymptotic tangents are the integral curves of these tangents. The asymptotic condition reads

$$\partial_{11}x, \partial_{22}x \in \operatorname{span}\{\partial_1 x, \partial_2\}. \tag{7}$$

A discrete version of this condition is the following: All symmetric 2^{nd} differences around a vertex are contained in the plane spanned by the first differences. This can be formulated in a symmetric way:

$$x, x_1, x_{\bar{1}}, x_2, x_{\bar{2}} \in \text{plane } P(u) \qquad \begin{array}{ccccc} & & x_2 & & \\ & & | & & \\ x_{\bar{1}} & — & x & — & x_1 \\ & & | & & \\ & & x_{\bar{2}} & & \end{array} \tag{8}$$

Consequently, discrete A-nets are defined by the requirement that all edges emanating from a vertex $x(u)$ must lie in a common plane $P(u)$.

3.2. Surfaces of constant Gaussian curvature. Surfaces of constant negative Gaussian curvature (*K-surfaces*) are interesting for a variety of reasons, e.g. because the geometry of their geodesics locally coincides with classical hyperbolic geometry. They are important for the development of discrete differential geometry, since they were among the first where a meaningful discretization has been obtained [**Sau50, Wun51**], and because of the integrable equations associated with K-surfaces.

Like any other negatively curved surface in Euclidean 3-dimensional space, a K-surface has an asymptotic parametrization. It is not difficult to show that for a smooth asymptotic surface, the additional *Chebyshev condition*

$$\partial_2\|\partial_1 x\| = \partial_1\|\partial_2 x\| = 0 \tag{9}$$

characterizes $K = \text{const}$. Since the length of the derivative vectors w.r.t. one variable does not depend on the other variable, we can re-parametrize and achieve

$$\|\partial_1 x\| = \|\partial_2 x\| = 1. \tag{10}$$

It turns out that in this case, when traversing a parameter line with velocity 1, the surface's normal vector is rotating with velocity $\sqrt{-K}$. An obvious discretization of the Chebyshev property is

$$\Delta_2 \|\Delta_1 x\| = \Delta_1 \|\Delta_2 x\| = 0. \tag{11}$$

The construction of discrete K-surfaces depends on the properties of a so-called Chebyshev quad, where opposite edges have equal lengths

$$\|x - x_1\| = \|x_2 - x_{12}\| \quad \text{and} \quad \|x - x_2\| = \|x_1 - x_{12}\| \tag{12}$$

and which does not lie in a plane (*skew parallelogram*). The following statement is elementary spatial geometry:

LEMMA 3.1. *A non-planar quadrilateral x—x_1—x_{12}—x_2 with the Chebyshev property given by* (12) *(opposite edges have equal lengths) is symmetric w.r.t. a 180° rotation about the axis spanned by the two midpoints of diagonals.*

In each vertex x we consider a unit normal vector n orthogonal to the edges incident with x. The cyclic orientation of the quadrilateral face defines the orientation of the normal vector. The twist angle along an edge which is enclosed by the planes at either end (resp. by their normal vectors) is the same for opposite edges,

$$\langle n, n_1 \rangle = \langle n_2, n_{12} \rangle, \quad \langle n, n_2 \rangle = \langle n_1, n_{12} \rangle$$

(the unit normal vectors constitute a Chebyshev quad). Further, we can reconstruct, up to scaling, the edges of the quadrilateral from these normal vectors by

$$x_1 - x = n \times n_1, \qquad x_2 - x = -n \times n_2.$$

Also the converse is true: Every net of unit normal vectors which fulfills these conditions determines a K-net.

For a proof we refer to [**BS09**, §4.2.1]. The equal angles property is an immediate corollary of the symmetry. An illustration in [**Wun51**] shows a K-net which has actually been built, and where both the Chebyshev property and the angle property can be observed.

The systematic treatment of A-nets and K-nets $x(u)$ relies on the construction of the so-called Lelieuvre normal vector field $m(u)$, which has the property $x_k - x = m_k \times m$. It is a so-called T-net, the T-nets being an integrable 3-system. In this way the geometric considerations of [**Sau50, Wun51**] concerning skew-parallelogram lattices can be treated in a more high-level manner. We summarize, without proof:

THEOREM 3.2. *A discrete K-net $x(i_1, \dots, i_d)$ (an asymptotic net with the Chebyshev property) has unit normal vectors $n(i_1, \dots . i_d)$ which form a Chebyshev net themselves: The rotation angle of the normal vector along an edge xx_k depends only on the k-th parameter, and not on the others.*

The K-net property is a 2-system in the following sense: If unit normal vectors $n(i_1, 0)$ and $n(0, i_2)$ are given, then these data can be uniquely extended to a Chebyshev net $n(i_1, i_2)$ in the unit sphere, and a K-net x is derived from n by the formulae above.

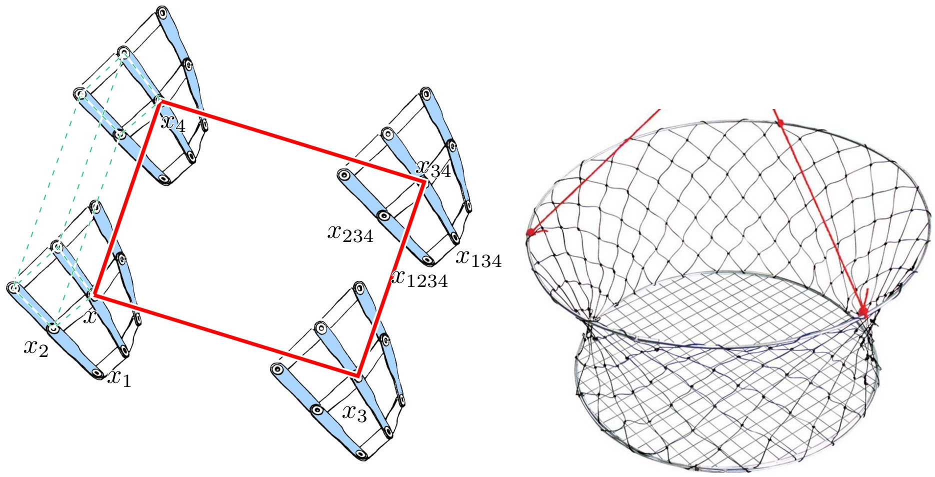


FIGURE 4. *Left:* A 4-dimensional discrete K-net, which at the same time is a 2D lattice of 2-dimensional K-nets, where each is in the Bäcklund transform relation with its neighbors. The thin dashed lines constitute a Bennett 12-bar linkage. *Right:* A K-net with rotational symmetry made from string (crab net).

The K-net property is d-dimensional consistent for $d \geq 3$ (i.e., integrable): Any choice of unit normal vectors

$$n(i_1, 0, \ldots, 0), \quad n(0, i_1, \ldots, 0), \quad \ldots, \quad n(0, 0, \ldots, i_d) \quad (d \geq 2)$$

can be extended to a Chebyshev net in the unit sphere, and define a K-net via the formulas given above.

3.3. Bäcklund transformation. Two smooth K-surfaces $x, x^{(1)}$, both parametrized such that parameter lines have unit speed, are Bäcklund transforms of each other, if the distance $\|x(u) - x^{(1)}(u)\|$ of corresponding points is constant, and if the line segment $x(u), x^{(1)}(u)$ is *tangent* to both surfaces in its endpoints. For a discrete surface $x(i_1, i_2)$ and its transform $x^{(1)}(i_1, i_2)$, the definition is literally the same, since even discrete K-nets possess tangent planes. Furthermore, it is clear from the definition that a K-net and its transform together constitute a 3-dimensional K-net.

Existence of 4D K-nets and the fact that they are determined by initial values (Theorem 3.2) shows Bianchi permutability of Bäcklund transforms, see Fig. 4.

3.4. Mechanisms based on skew parallelograms. The results on K-nets have as an immediate consequence the following remarkable fact which was first observed, at least in part, more than 100 years ago by G. T. Bennett [**Ben14**]:

Think of the K-net defined by normal vector data $n(i_1, 0)$ and $n(0, i_2)$. Deforming these initial values such that the angle between successive normal vectors remains constant yields a net $n(i_1, i_2)$ of normal vectors where the angle between neighbors remains constant. Computing the K-net from normal vector data yields a deformation of $x(i_1, i_2)$ such that both edge lengths and the twist along an edge remains constant.

Already a single quadrilateral is a nontrivial mechanism, called Bennett's four-bar mechanism or Bennett's isogram [**Ben14**]. An elementary cube in a 3-dimensional K-net is Bennett's 12-bar mechanism, see also Figure 4.

It is worth noting that certain K-nets can be manufactured rather easily, namely as equilibrium positions of fishing nets with rotational symmetry, see the crab net shown by Figure 4.

3.5. The sine-Gordon equation. By Gauss' *theorema egregium*, the Gaussian curvature K of a surface $x(u_1, u_2)$ can be computed from the coefficient functions $E = \langle \partial_1 x, \partial_1 x\rangle$, $F = \langle \partial_1 x, \partial_2 x\rangle$ and $G = \langle \partial_2 x, \partial_2 x\rangle$ of the first fundamental form. Francesco Brioschi's formula for Gaussian curvature says that

$$\begin{vmatrix} -\partial_{22}E + 2\partial_{12}F - \partial_{11}G & \partial_1 E & 2\partial_1 F - \partial_2 E \\ 2\partial_2 F - \partial_1 G & 2E & 2F \\ \partial_2 G & 2F & 2G \end{vmatrix} - \begin{vmatrix} 0 & \partial_2 E & \partial_1 G \\ \partial_2 E & 2E & 2F \\ \partial_1 G & 2F & 2G \end{vmatrix} = 8(EG - F^2)^2 K.$$

The unit speed (Chebyshev) property is expressed as $E = G = 1$, and with the angle $\phi(u_1, u_2)$ between parameter lines we have $F = \cos\phi$. Brioschi's formula yields $\partial_{12}\phi(u_1, u_2) = -K(u_1, u_2) \cdot \sin\phi(u_1, u_2)$. In the case of K-surfaces, it is no loss of generality to assume $K = -1$, so

$$\partial_{12}\phi = \sin\phi. \tag{13}$$

The meaning of this *sine-Gordon equation* equation in terms of the intrinsic geometry of a surface is this: If a surface is parametrized such that parameter lines are traversed with unit speed, and the angle between parameter lines evolves according to the sine-Gordon equation, then the surface has constant Gaussian curvature -1.

Note that Eqns. (10)+(13) constitute a purely intrinsic characterization of surfaces with $K = -1$, while Eqns. (10)+(7) provide an extrinsic characterization.

3.6. Discrete sine-Gordon equation. Integrable equations. A discrete K-surface where all edge lengths are equal to ε is the most direct analogue of a smooth surface parametrized by unit speed with Gaussian curvature -1. The evolution of the angle ϕ enclosed by edges xx_1 and xx_2 can be shown to obey the Hirota equation

$$\sin\frac{\phi_{12} - \phi_1 - \phi_2 + \phi}{4} = \frac{\varepsilon^2}{4}\sin\frac{\phi_{12} + \phi_1 + \phi_2 + \phi}{4}. \tag{14}$$

In these lecture notes we are not able to discuss the term *integrability* in a systematic manner, we refer to [**BS09**, §6] instead. We only mention some salient facts:

- integrable equations turn out to be closely related with multidimensionally consistent geometric properties of nets;
- discrete nets are easier to treat than continuous ones, in particular the special role of transformations disappears. The smooth case is obtained from the discrete case by a limit process;
- typical properties of integrable systems (Bäcklund transforms, zero curvature representations, ...) are a consequence of multidimensional consistency.

4. Applications: computing minimal surfaces

There is a nice version of discrete minimal surfaces which connects an analytic approach with discrete geometry and circle patterns. It is also capable of solving a significant problem, namely computing the shape of a minimal surface from the combinatorics of its principal curve network [**BHS06**].

4.1. Isothermic surfaces and their duals. Isothermic surfaces represent a classical topic of differential geometry. Informally they are defined by the condition that their infinitesimal quadrilaterals are flat squares. Certain surface classes, including the minimal surfaces, admit an isothermic parametrization. It is a bit of a mystery why certain classes of surfaces have this property. For a good overview see [**Bur06**].

THEOREM 4.1. *An isothermic surface $x : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, characterized by the condition that it is a conformal curvature-line parametrization, i.e.,*

$$\|\partial_1 x\| = \|\partial_2 x\| = s, \quad \langle \partial_1 x, \partial_2 x\rangle = 0, \quad \partial_{12} x \in \operatorname{span}(\partial_1 x, \partial_2 x),$$

with $s : U \to \mathbb{R}_+$, has a Christoffel-dual surface x^* defined by

$$\partial_1 x^* = \frac{1}{s^2}\partial_1 x \quad \partial_2 x^* = -\frac{1}{s^2}\partial_2 x. \tag{15}$$

x^* is again isothermic, with $\|\partial_j x^*\| = 1/s$.

PROOF. We have $\partial_{12}x = a\partial_1 x + b\partial_2 x$, for certain coefficient functions a, b. Further, $\partial_2(s^2) = 2\langle \partial_1 x, \partial_{12}x\rangle = 2\langle \partial_1 x, a\partial_1 x + b\partial_2 x\rangle = 2as^2$. Differentiating $\partial_1 x^*$ yields

$$\partial_2(\partial_1 x^*) = -\frac{\partial_2(s^2)}{s^4}\partial_1 x + \frac{1}{s^2}\partial_{12}x = -\frac{2a}{s^2}\partial_1 + \frac{1}{s^2}(a\partial_1 x + b\partial_2 x) = \frac{1}{s^2}(b\partial_2 x - a\partial_1 x).$$

For $\partial_1(\partial_2 x^*)$ an analogous computation yields the same result, so x^* exists. Obviously, x^* fulfills the isothermicity conditions. □

The Christoffel transformation is particularly interesting for *minimal surfaces*, since they admit isothermic parameters, and they are Christoffel-dual to their own unit normal vector field (which is then a conformal parametrization of the unit sphere). Conversely, any conformal parametrization of the unit sphere by Christoffel duality is converted into a minimal surface. If the unit sphere is identified with $\mathbb{C} \cup \{\infty\}$ by stereographic projection, and the definition of the Christoffel dual is written as an integral, this yields the Weierstrass representation of minimal surfaces.

Some parts of this relationship are easy to show. E.g. if x is a conformal parametrization of the unit sphere, then x is isothermic. The common unit normal vector field of x and the Christoffel dual x^* is furnished by x itself. Compute the principal curvatures of x^*:

$$\kappa_1 = \frac{\partial_1 x}{\partial_1 x^*} = s^2, \quad \kappa_2 = \frac{\partial_2 x}{bd_2 x^*} = -s^2.$$

i.e., x^* is a minimal surface. We also see how the speed of parametrization (which is s) is related to the values of the principal curvatures.

4.2. Discrete s-isothermic surfaces. Finding the "right" discretization of isothermic surfaces is an important topic in discrete differential geometry. The discretization shown here is one example of several – it is selected because of its success in solving a continuous problem by discretization. We start this discussion with a remarkable result on polyhedra. It can be seen as a discrete version of the Koebe uniformization theorem in the genus 0 case, and it can be made computationally efficient [**Sech07**].

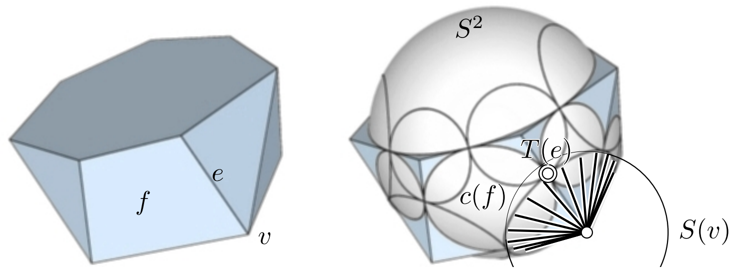


FIGURE 5. A Koebe polyhedron has edges tangent to a sphere. The orthogonal circle pattern defined by this polyhedron consists of the incircles of faces, and the circles of tangency of cones whose vertex is a vertex of the polyhedron. The vertex-centered sphere used in the definition of an s-isothermic surface is also shown.

THEOREM 4.2. *For each convex polyhedron there is a combinatorially equivalent convex "Koebe" polyhedron whose edges are tangent to the unit sphere. It becomes unique up to Euclidean transformations if we require that its center of mass equals* 0*, otherwise it is unique only up to Möbius transforms (i.e., up to projective automorphisms of the unit sphere).*

Each Koebe polyhedron has two associated circle packings, see Fig. 5. System (i) arises by intersecting the faces' planes with the sphere; System (ii) arises in a dual way as the circles of tangency of cones whose vertex is a vertex of the polyhedron. System (ii) can also be replaced by vertex-centered spheres, showing that a Koebe polyhedron fulfills the definition of a discrete s-isothermic surface, see Fig. 6.

DEFINITION 4.3. *A polyhedral surface (in particular a discrete conjugate net) is s-isothermic, if the following conditions are fulfilled.*

- *Every face* f *contains an incircle* $c(f)$*, every vertex* v *is the center of a sphere* $S(v)$*.*

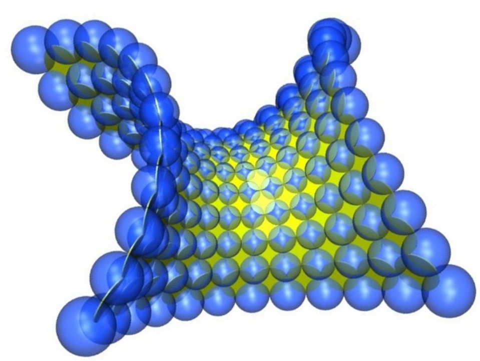


FIGURE 6. An s-isothermic surface is defined as a circle/sphere arrangement where circles (green) touch circles, spheres (blue) touch spheres, and circles orthogonally intersect spheres. This example is a discrete Enneper surface (image courtesy T. Hoffmann).

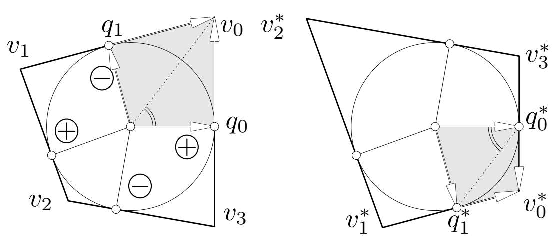
FIGURE 7. Attempting to dualize a piece of Koebe polyhedron where one face has odd valence. The dual of the triangle does not close up (we cannot even assign labels ⊕,⊖ to edges in a consistent manner). However, if the given discrete surface is seen as a discrete branched covering, with the triangle actually being a hexagon, the dual surface will close after the original one is traversed twice (images courtesy B. Springborn).

- *Every edge e contains a point $T(e)$, such that:*
  (*) $v \in e \implies S(v)$ *intersects e orthogonally in $T(e)$*
  (*) $e \subset f \implies c(f)$ *touches e in $T(e)$*
- *Vertices have degree four, and faces have even degree.*

If in addition the surface is part of a Koebe polyhedron, i.e., its edges are tangent to S^2, then we regard it as a discrete-conformal parametrization of S^2.

4.3. Dualizing s-isothermic surfaces. Consider a face $f = (v_0, \dots, v_{n-1})$ of an s-isothermic surface with n even. Assume that the incircle of this face has radius ρ_f. The edge $v_i v_{i+1}$ touches the incircle in the point q_i. Then the dual face f^* likewise has an incircle of radius $\rho_{f^*} = 1/\rho_f$, and the new contact points q_i^* are defined by

(16)



$$q_i^* = (-1)^i \frac{q_i}{\rho_f^2}$$

It follows that the edges of the dual and primal polygon are related by

$$p_j^* - p_{j+1}^* = (-1)^j \frac{1}{r_j r_{j+1}} (p_i - p_{i+1}), \text{ where } r_j = \|v_j - q_j\| = \|v_j - q_{j+1}\|. \tag{17}$$

The values r_j are the radii of the vertex-centered spheres which occur in the definition of an s-isothermic surface. The proof is an exercise in complex numbers. Similarly, one can show that the subdivision of the faces into quads as in (16) yields corresponding discrete surfaces $x(i_1, i_2)$ and $x^*(i_1, i_2)$ where

$$\Delta x_1^* = \frac{1}{\|\Delta x_1\|^2} \Delta x_1, \quad \Delta x_2^* = -\frac{1}{\|\Delta x_2\|^2} \Delta x_2. \tag{18}$$

This is completely analogous to the smooth case of (15).

For any given s-isothermic surface, we now assign labels $\oplus$ and $\ominus$ to all edges, such that for each face, the cycle of edges is assigned labels $\oplus$, $\ominus$, $\oplus$, $\ominus, \dots$ in an alternating way, indicating if the factor $+1$ or -1 is to be used in (16). By (17), the edge lengths of dual edges are independent of the face they are contained in. An interior angle $\alpha_{v,f}$ at the vertex v in the face f corresponds to the $\alpha^*_{v^*,f^*} = \pi - \alpha_{v,f}$ in the dual face, so the angle sum around the vertex v^* is

$$\sum_{f^* \ni v^*} \alpha^*_{v^*,f^*} = \sum_{f \ni v} (\pi - \alpha_{v,f}) = (\deg(v) - 2)\pi = 2\pi.$$

This shows that a discrete s-isothermic surface can be dualized.

Definition 4.4. *A discrete s-isothermic surface is minimal, if its dual is spherical, i.e., is part of a Koebe polyhedron.*

[**BHS06**] discusses how to compute the shape of a minimal surface from the combinatorics of its network of principal curvatures lines. One starts by computing a Koebe polyhedron with the desired combinatorics, and dualizes, see Fig. 7. One can even show convergence to smooth minimal surface, by using the fact that correspondences between combinatorially equivalent circle packings can be shown to converge to conformal mappings [**HS93**].

4.4. Curvatures of discrete surfaces. Discrete surfaces have been assigned curvatures in different ways. E.g. the variational definition of the mean curvature vector field $\vec{H}$ as the gradient of the area functional leads to *cotangent formula* for mean curvature of simplicial surfaces. Another approach uses the *Steiner formula*: A surface in $\mathbb{R}^3$, with unit normal vector n, has constant-distance offset surfaces where a point p of the original surface moves to $p + tn(p)$. Surface area changes according to

$$A^t = \int (1 - 2Ht + Kt^2)\, dA, \tag{19}$$

where dA refers to the canonical surface area measure, and H, K are mean curvature and Gaussian curvature, respectively.

[**PLW$^+$07, BPW10**] proposed to use the same principle for a polyhedral surface (V, E, F) equipped with a polyhedral Gauss image $(\sigma(V), \sigma(E), \sigma(F))$ (i.e., normal vector field), such that corresponding faces of surface and Gauss image are parallel. An example is furnished by the s-isothermic minimal surfaces and the corresponding Koebe polyhedron. We define a constant-distance offset (V^t, E^t, F^t) by vertices

$$v^t = v + t\sigma(v).$$

The oriented area of closed polygon $f = (v_0, \dots, v_{n-1})$ in $\mathbb{R}^2$ is a quadratic form $A(f)$ with associated symmetric bilinear form $A(f, g)$,

$$A(f) = \frac{1}{2} \sum\nolimits_{i=0}^{n-1} \det(v_i, v_{i+1}), \quad A(f, g) = \frac{1}{2}(A(f + g) - A(f) - A(g))$$

(Leibniz' sector formula). $A(f, g)$ is called the oriented mixed area of f, g. Then the area of a face $f^t \in F^t$ is given by

$$A(f^t) = A(f) + 2tA(f, \sigma(f)) + t^2 A(\sigma(f)). \tag{20}$$

Comparing (19) with (20) leads to the definition of a mean curvature H and a Gaussian curvature K of a face f:

$$H(f) = -\frac{A(f, \sigma(f))}{A(f)}, \quad K(f) = -\frac{A(\sigma(f)}{A(f)}. \tag{21}$$

With this definition it is not difficult to compute the mean curvature of s-isothermic surfaces: it turns out to be zero. The considerations above have been extended to more general surface classes by [**HSFW17**]. E.g the discrete K-surfaces indeed enjoy the property that the Gaussian curvature equals -1.

5. Applications: freeform architecture

With computer-aided design, it has become rather easy for professionals to design free forms in architecture, but building them is another matter. The geometric questions which arise in this context have a rich connection to discrete differential geometry. Recent surveys are [**PW16, PEVW15**]. Examples of specific geometric issues in the context of freeform architecture are the following.

- Steel-glass constructions often are discrete surfaces with flat faces, because of the glass panels that are put there. For surfaces with quadrilateral faces, the planarity requirement is a nontrivial side-condition. Another issue with steel constructions is the manner of intersection of the beams in nodes. We discuss this topic below, see also [**LPW$^+$06, PLW$^+$07**].
- Freeform skins might be required to be at constant distance from each other, or a steel-glass construction might be required to be of constant thickness. This topic leads to circular nets, conical nets, and even nets which are edgewise parallel to a Koebe polyhedron, depending on the question how distances are measured (between vertices, or faces, or edges), see [**PLW$^+$07**].
- Developable surfaces occur in bent glass and in curved beams (whose sides are made by bending flat pieces of steel). A sequence of developables is a semidiscrete conjugate net, a viewpoint which has been exploited by [**LPW$^+$06, PSB$^+$08**].
- The differential geometry of manifolds in line space, and discrete submanifolds of line space has been used by [**WJB$^+$13**] to study lighting and shading.
- Circle packings and discrete uniformization turn up in the question of regular patterns, in particular hexagonal patterns [**SHWP09**].
- Self-supporting surfaces like brick vaults and their polyhedral Airy potential surface are related to isotropic geometry and curvatures [**VHWP12**]. A discrete stress state and corresponding polyhedral Airy potential is also relevant for material-minimizing structures [**PKWP17**].

5.1. Meshes. Torsion-free support structures. Here we discuss briefly some topics from the list which have to do with discrete *parametric* surfaces. In order to describe structures from straight or curved beams, with or without panels covering the faces, we start with the combinatorics. A mesh (V, E, F) consists of a graph (V, E), where individual edge cycles are designated as faces $f \in F$. This has to be done in such a way that the complex glued from points (corresponding to vertices), line segments (corresponding to edges) and disks (one for each face) is a topological manifold. In a geometric realization of the connectivity, edges are realized as line segments or as curves. Faces are filled by planar or non-planar surfaces.

Figure 8. This torsion-free support structure (image courtesy Evolute GmbH) guides members and nodes in the outer hull of the Yas Marina hotel in Abu Dhabi, so that members have a nice intersection in each node (at right, image courtesy Waagner-Biro Steel and Glass). Note that the faces of this mesh are not planar.

Many freeform architectural structures can be modelled by means of meshes, see e.g. Figure 8 for a mesh where faces are not filled in, but edges are represented by straight beams and vertices by connectors between the individual beams.

If faces are combinatorial triangles, resp. quads or hexagons, we speak of a triangle mesh, resp. quad mesh or hexagonal mesh. Terminology is often sloppy and one applies the term quad mesh or hex mesh also in cases where most faces are quads, or most are hexagons. The discrete parametric surfaces mapping from $\mathbb{Z}^2$ to $\mathbb{R}^3$ are quad meshes in a natural way.

Definition 5.1. *A torsion-free support structure associated to a mesh* (V, E, F) *with straight edges is an assignment of a line* $\ell(v)$ *to each vertex* $v \in V$ *and a plane* $\pi(e)$ *to each edge* $e \in E$*, such that* $v \in e \implies \ell(v) \subset \pi(e)$ *(plus certain nondegeneracy conditions).*

If the edges of the mesh are curves, $\pi(e)$ *is a developable surface containing the edge.*

The relevance of a torsion-free support structure is that straight beams in a steel construction can be aligned with the planes $\pi(e)$, and the symmetry planes of beams nicely intersect in the line $\ell(v)$, whenever beams come together in a node, see Fig. 8. Such an intersection of beams is called torsion-free. If the intersection is not torsion-free, and beams intersect anyhow, the intersection is complicated to manufacture, see Fig. 9. However, if the intersection is torsion-free, one can simply use a cylindrical node element and connect the beams to it.

Proposition 5.2. *A properly connected triangle mesh has only trivial support structures, where all elements pass through a single center* z*: for all* $v \in V, e \in E$ *we have* $\ell(v) = v \vee z$ *and* $\pi(e) = e \vee z$*.*

Proof. In projective space, and ignoring degeneracies, we argue as follows. In a face $f = v_i v_j v_k$, we have $\ell(v_i) = \pi(v_i v_j) \cap \pi(v_i v_k)$, and similar for $\ell(v_j)$, $\ell(v_k)$. Thus, the point $z(f) = \pi(v_i v_j) \cap \pi(v_i v_k) \cap \pi(v_j v_k)$ lies on all lines $\ell(v_i)$, $\ell(v_j)$, $\ell(v_k)$. If $f' = (v_i v_j v_l)$ is a neighbor face, then $z(f) = \ell(v_i) \cap \ell(v_j) = z(f')$. By connectedness, $z(f)$ is the same point for all faces $f \in F$. $\square$

This result is one reason why quad meshes are attractive for steel-glass constructions. Unfortunately, imposing planarity of faces on quad meshes is nontrivial.

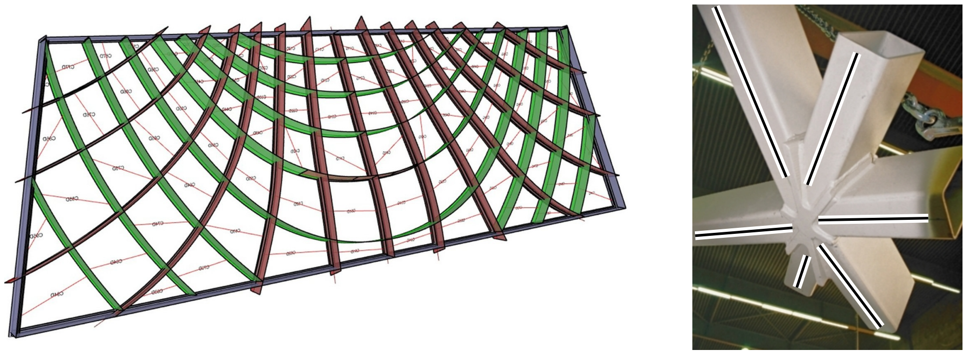


FIGURE 9. *Left:* A curved support structure (Eiffel tower pavilions, Paris. Image Evolute GmbH, 2010). *Right:* An intersection of beams which does not follow a torsion-free support structure (courtesy Waagner Biro Steel and Glass).

5.2. The design dilemma. When using mathematical methods in an artistic context, one faces a problem which is not known from applications in the natural sciences: It is usually not desirable that Mathematics yields a definite answer to a certain question. We illustrate this by means of torsion-free support structures: Such a structure either consists of developable surfaces, or of discrete developables following the edges of the mesh. If they are to be *orthogonal* to the reference surface, like in the case of Figure 9, left, then they must follow the surface's principal curvature lines, cf. the discussion at the end of §2. Since the principal curvature lines are determined by the reference surface, there is no longer any design freedom for the beams except perhaps their spacing. Such a situation, which amounts to a restriction of the artist's freedom of expression, is unacceptable to the designer. In the case of the Eiffel tower pavilions, design freedom was restored by the fact that small changes to the design surface can cause big changes in the network of principal curvature lines. In this way it was possible to achieve the desired layout of beams by only minimally changing the reference surface.

5.3. Meshes with planar faces – conjugate nets. The design of quadrilateral meshes with planar faces was among the first topics where a connection between freeform architecture and discrete differential geometry was established [**LPW$^+$06**]. If the mesh under consideration has the visual appearance of a smooth network of curves on a smooth surface, then such a quad mesh has mostly regular combinatorics and is therefore a conjugate net which approximates a conjugate curve network, see Fig. 2. If in addition to conjugacy we add orthogonality of curves, we have the principal network. In fact, any conjugate network useful for deriving a quad mesh from it is close to principal, which is another instance of the design dilemma mentioned above. Since the reference shape and a conjugate net on it cannot be designed separately, there are very few instances of such meshes which serve as basis of actual building skins, and most of those have been generated by a simple process like translating one polygon along another one, see Fig. 10.

FIGURE 10. This quad mesh with parallelogram faces has been constructed by parallel-translating a polygon along another polygon (Hippo house, Berlin Zoo. Engineering by Schlaich Bergermann & Partners).

5.4. Constant-distance offsets. In the context of meshes with planar faces, a natural question is the existence of another mesh which is combinatorially equivalent and which is *at constant distance* from the first one. One can consider face-face distances, or edge-edge distances, or vertex-vertex distancesconstant. In order to treat these cases in an uniform manner, we first define parallelity of meshes:

DEFINITION 5.3. *Consider combinatorially equivalent meshes* (V, E, F) *and* (V', E', F') *with straight edges and planar faces. They are parallel, if corresponding edges of* E *resp.* E' *lie in parallel lines, and corresponding faces of* F *resp.* F' *lie in parallel planes (details regarding nondegeneracy have been omitted).*

A conjugate net and its parallel net together constitute a 3D conjugate net. It is not difficult to show the following [**PLW**$^+$**07**]:

• If (V', E', F') is parallel to (V, E, F), then a torsion-free support structure is defined by $\ell(v) = v \vee v'$ and $\pi(e) = e \vee e'$, where v, v' and e, e' are pairs of corresponding vertices resp. edges. In the simply connected case, a torsion-free support structure also implies existence of parallel meshes

• Parallel meshes which are at constant vertex-vertex distance d in a nontrivial way are circular. This is because we can construct a third mesh by computing vertices according to $v'' = (v' - v)/d$, which is inscribed to the unit sphere and has planar faces, so it is circular. For a quadrilateral, circularity is expressed via the angles between edges only, so the two meshes we start with inherit the circular property.

These two statements show that conjugate nets and circular nets we studied in the beginning, have actual applications in a field which generally was not known for using much applied Mathematics at all, namely the design of freeform skins for architecture.

Acknowledgments. The author is supported by by SFB-Transregio Program *Discretization in Geometry and Dynamics*, Austrian Science Fund grant no. I2978.

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