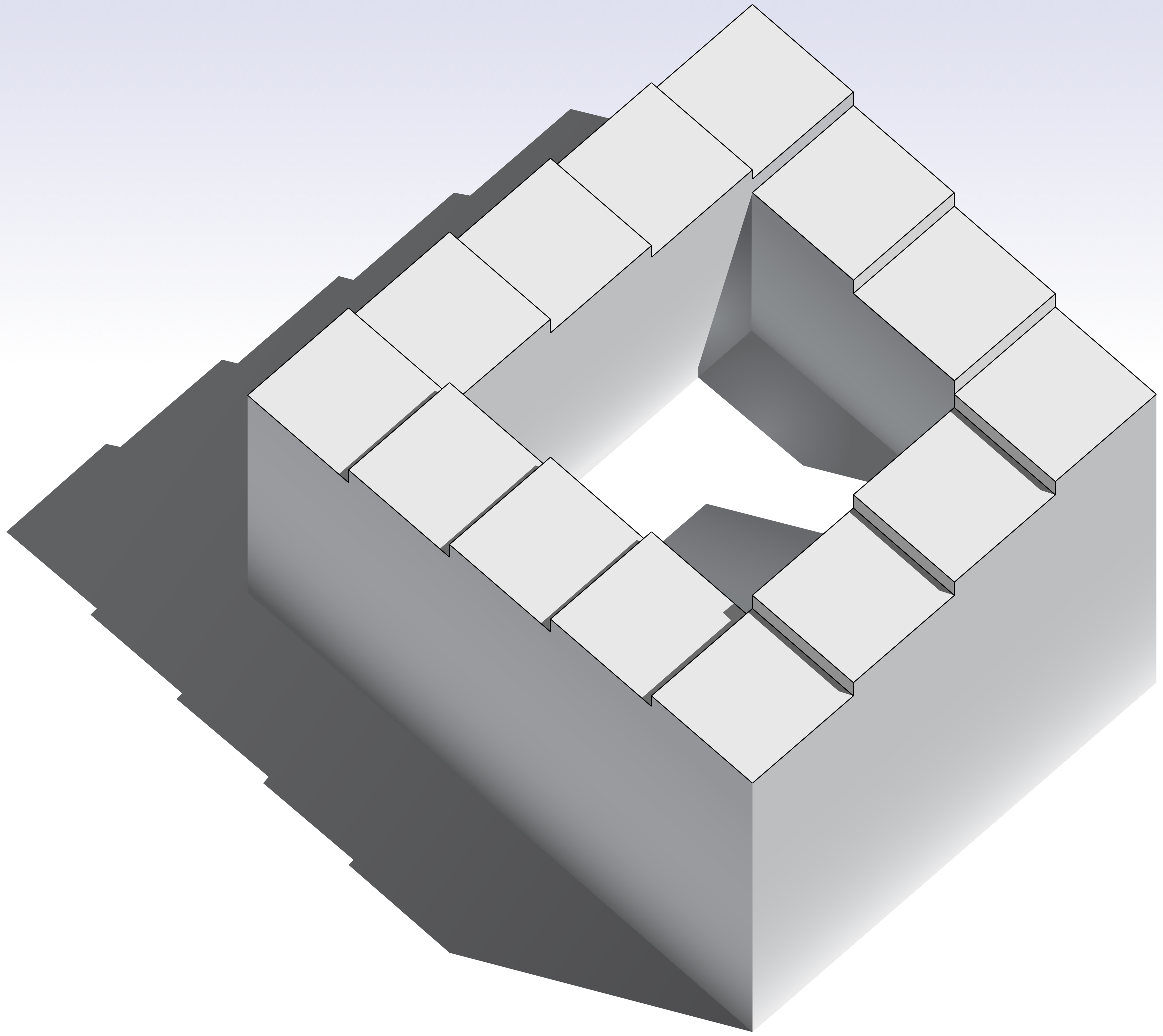
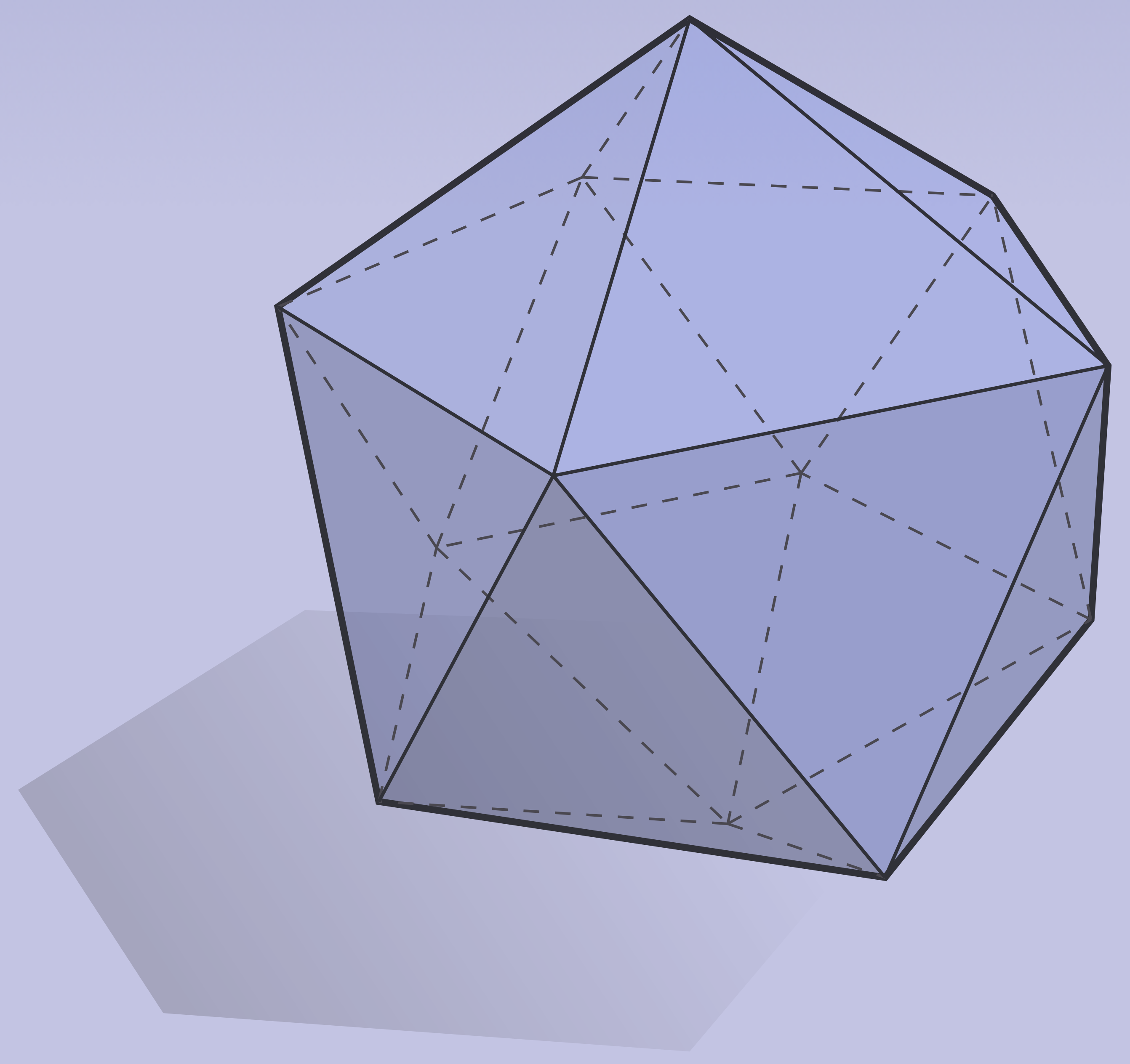


AMS Short Course

Discrete Differential Geometry

Joint Mathematics Meeting • San Diego, CA • January 2018

Course Overview



AMS Short Course

Discrete Differential Geometry

Joint Mathematics Meeting • San Diego, CA • January 2018

Welcome!

Schedule

Day 1	8am–8:30am	Introduction & Overview
	11:10am–12:10pm	Discrete Parametric Surfaces I
	9:30am–9:50am	*(break)*
	1:30pm–2:30pm	Discrete Parametric Surfaces II
	10:50am–11:10am	*(break)*
	8:30am–9:30am	Discrete Laplace Operators I
	12:10pm–1:30pm	*(lunch)*
	9:50–10:50am	Discrete Laplace Operators II
	2:30pm–3:30pm	*(free time)*
	3:30pm–5:00pm	Demo Session

Day 2	8:00am–9:00am	Discrete Mappings I
	9:00am–9:20am	*(break)*
	9:20am–10:20am	Discrete Mappings II
	10:20am–10:40am	*(break)*
	10:40am–11:40am	Discrete Conformal Geometry I
	11:40am–1:00pm	*(lunch)*
	1:00pm–2:00pm	Discrete Conformal Geometry II
	2:00pm–2:20pm	*(break)*
	2:20pm–3:20pm	Optimal Transport on Discrete Domains I
	3:20pm–3:50pm	*(break)*
	3:50pm–4:50pm	Optimal Transport on Discrete Domains II
	4:50pm–5:00pm	Wrap-up

Short Course Speakers

Johannes Wallner (TU Graz)
Discrete Parametric Surfaces

Max Wardetzky (Göttingen)
Discrete Laplace Operators

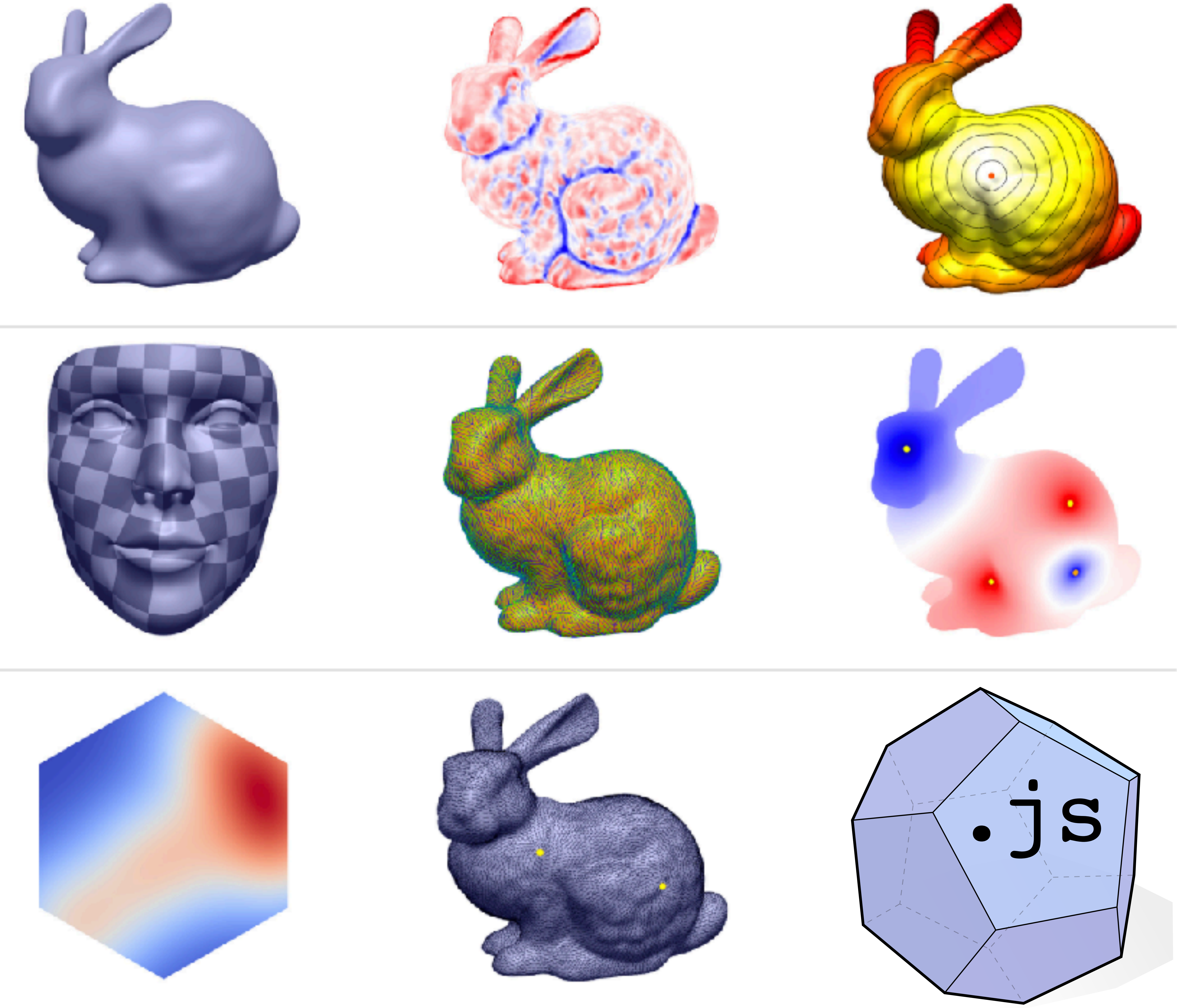
Yaron Lipman (Weizmann)
Discrete Mappings

***Keenan Crane (Carnegie Mellon)**
Discrete Conformal Geometry

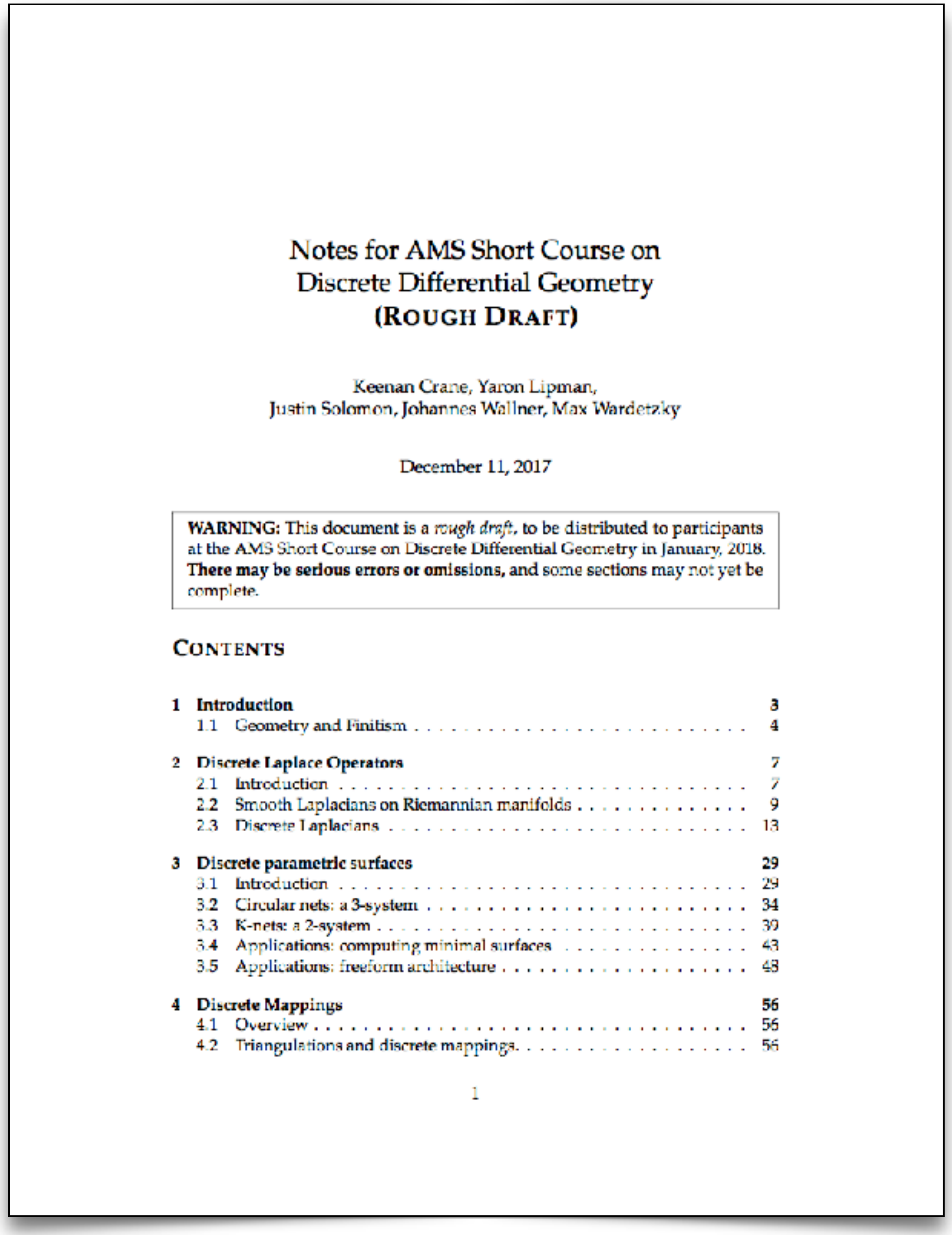
Justin Solomon (MIT)
Discrete Optimal Transport

Demo Session

- **Goal:** give participants hands-on experience w/ DDG algorithms
- <u>Implement</u> (in web-based framework):
  - *discrete curvature*
  - *discrete Laplace-Beltrami*
- <u>Experiment:</u>
  - geodesic distance
  - direction fields
  - conformal mapping
  - …



Reading Material



Notes for AMS Short Course on
Discrete Differential Geometry
(Rough Draft)

Keenan Crane, Yaron Lipman,
Justin Solomon, Johannes Wallner, Max Wardetzky

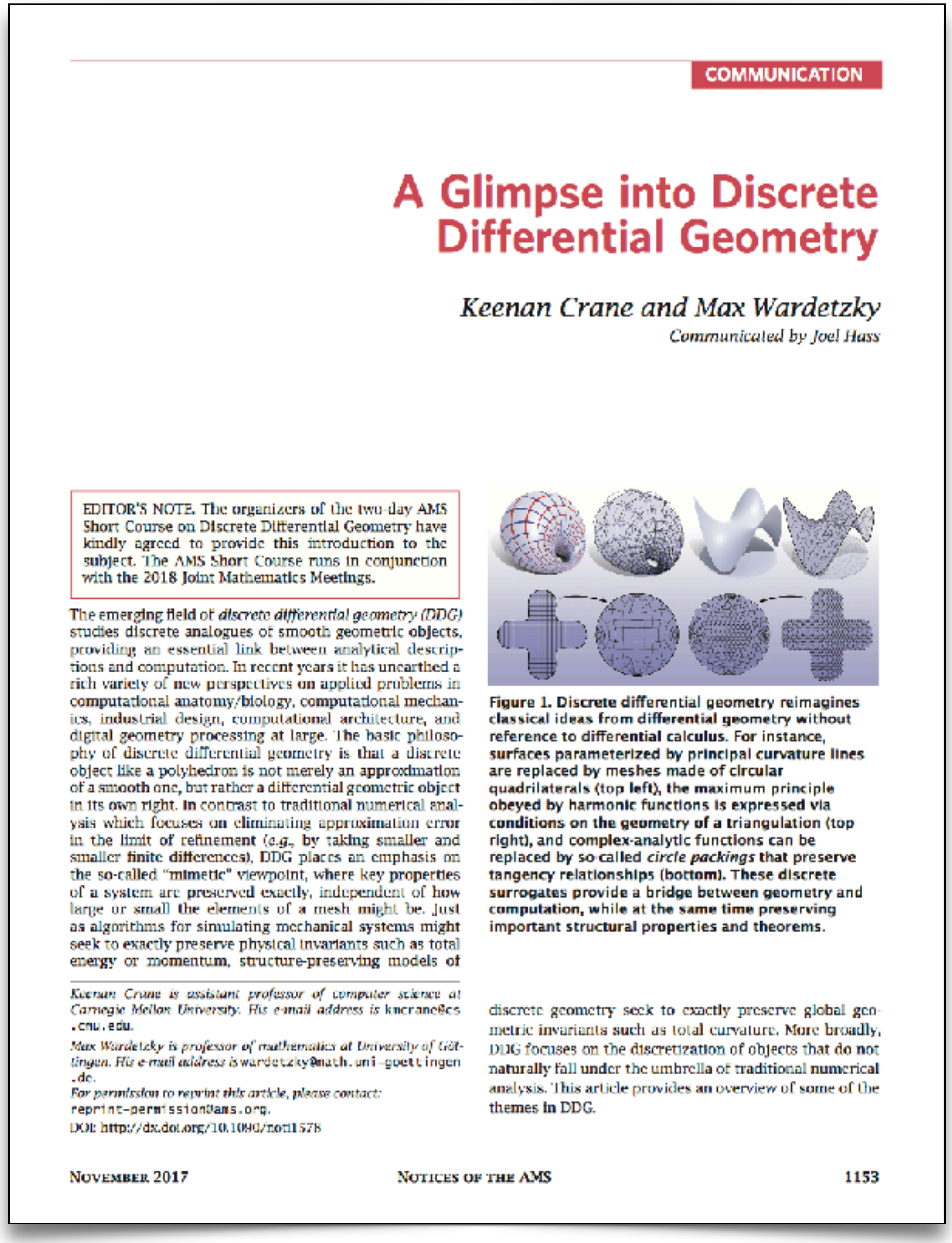
December 11, 2017

WARNING: This document is a *rough draft*, to be distributed to participants at the AMS Short Course on Discrete Differential Geometry in January, 2018. There may be serious errors or omissions, and some sections may not yet be complete.

Contents

1

Short Course Notes



COMMUNICATION

A Glimpse into Discrete Differential Geometry

Keenan Crane and Max Wardetzky
Communicated by Joel Hass

EDITOR'S NOTE. The organizers of the two-day AMS Short Course on Discrete Differential Geometry have kindly agreed to provide this introduction to the subject. The AMS Short Course runs in conjunction with the 2018 Joint Mathematics Meetings.

The emerging field of *discrete differential geometry (DDG)* studies discrete analogues of smooth geometric objects, providing an essential link between analytical descriptions and computation. In recent years it has unearthed a rich variety of new perspectives on applied problems in computational anatomy/biology, computational mechanics, industrial design, computational architecture, and digital geometry processing at large. The basic philosophy of discrete differential geometry is that a discrete object like a polyhedron is not merely an approximation of a smooth one, but rather a differential geometric object in its own right. In contrast to traditional numerical analysis which focuses on eliminating approximation error in the limit of refinement (*e.g.*, by taking smaller and smaller finite differences), DDG places an emphasis on the so-called "mimetic" viewpoint, where key properties of a system are preserved exactly, independent of how large or small the elements of a mesh might be. Just as algorithms for simulating mechanical systems might seek to exactly preserve physical invariants such as total energy or momentum, structure-preserving models of discrete geometry seek to exactly preserve global geometric invariants such as total curvature. More broadly, DDG focuses on the discretization of objects that do not naturally fall under the umbrella of traditional numerical analysis. This article provides an overview of some of the themes in DDG.

Figure 1. Discrete differential geometry reimagines classical ideas from differential geometry without reference to differential calculus. For instance, surfaces parameterized by principal curvature lines are replaced by meshes made of circular quadrilaterals (top left), the maximum principle obeyed by harmonic functions is expressed via conditions on the geometry of a triangulation (top right), and complex-analytic functions can be replaced by so-called *circle packings* that preserve tangency relationships (bottom). These discrete surrogates provide a bridge between geometry and computation, while at the same time preserving important structural properties and theorems.

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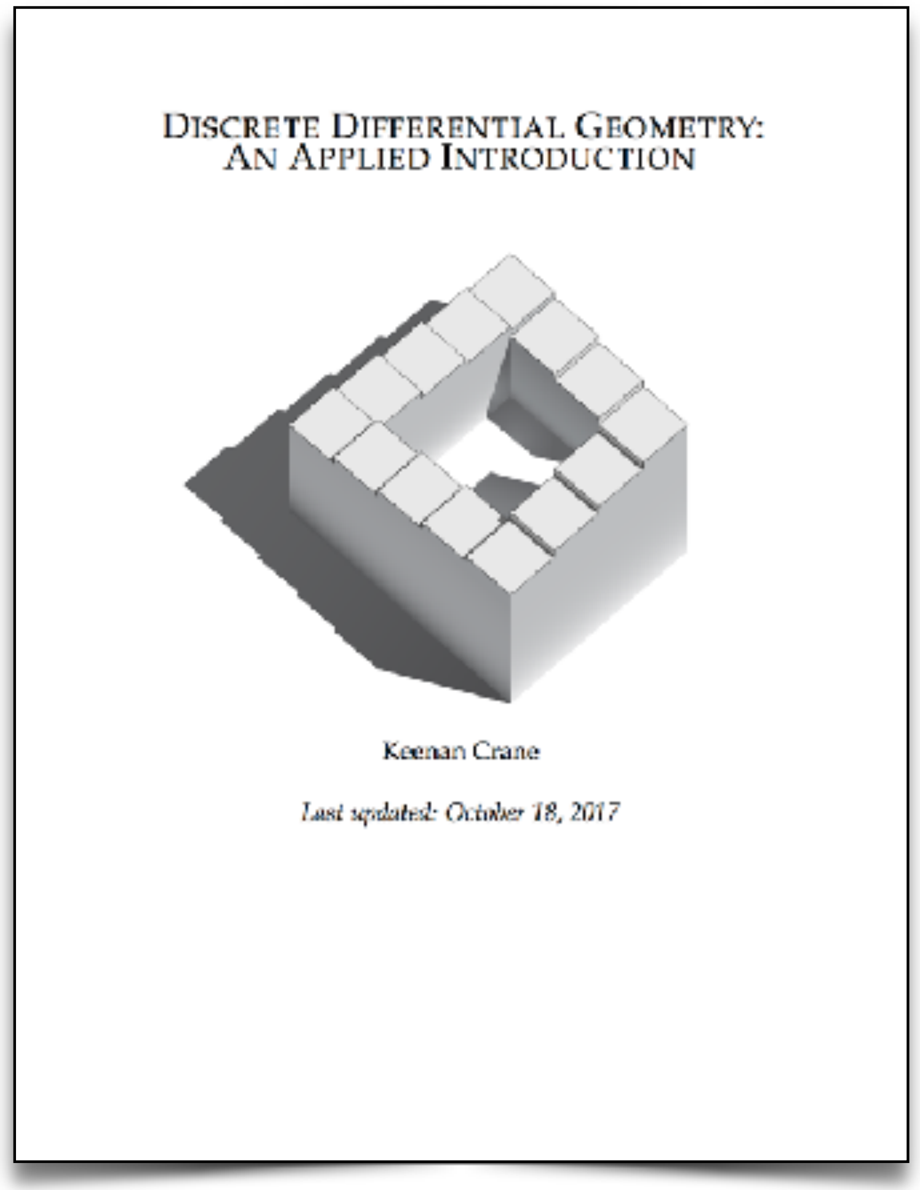
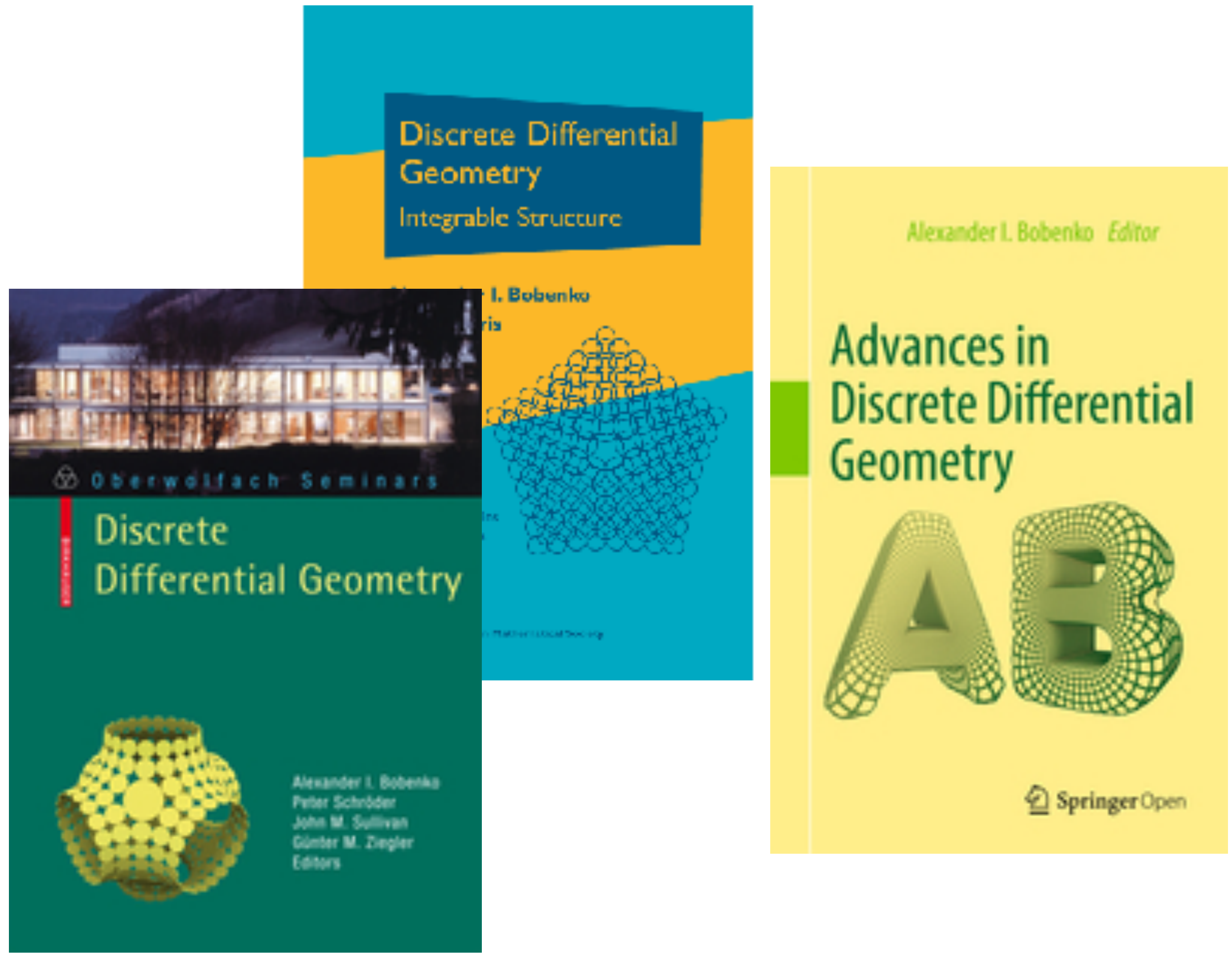
AMS Notices Article

Want to Know More?

- Several books:
  - *Discrete Differential Geometry* (2008)
  - *DDG: Integrable Structure* (2008)
  - *Advances in Discrete Differential Geometry* (2016)
  - *Architectural Geometry* (2007)
- CMU Course: (http://geometry.cs.cmu.edu/ddg)
- More links at *Discrete Differential Geometry Forum:*

http://ddg.cs.columbia.edu





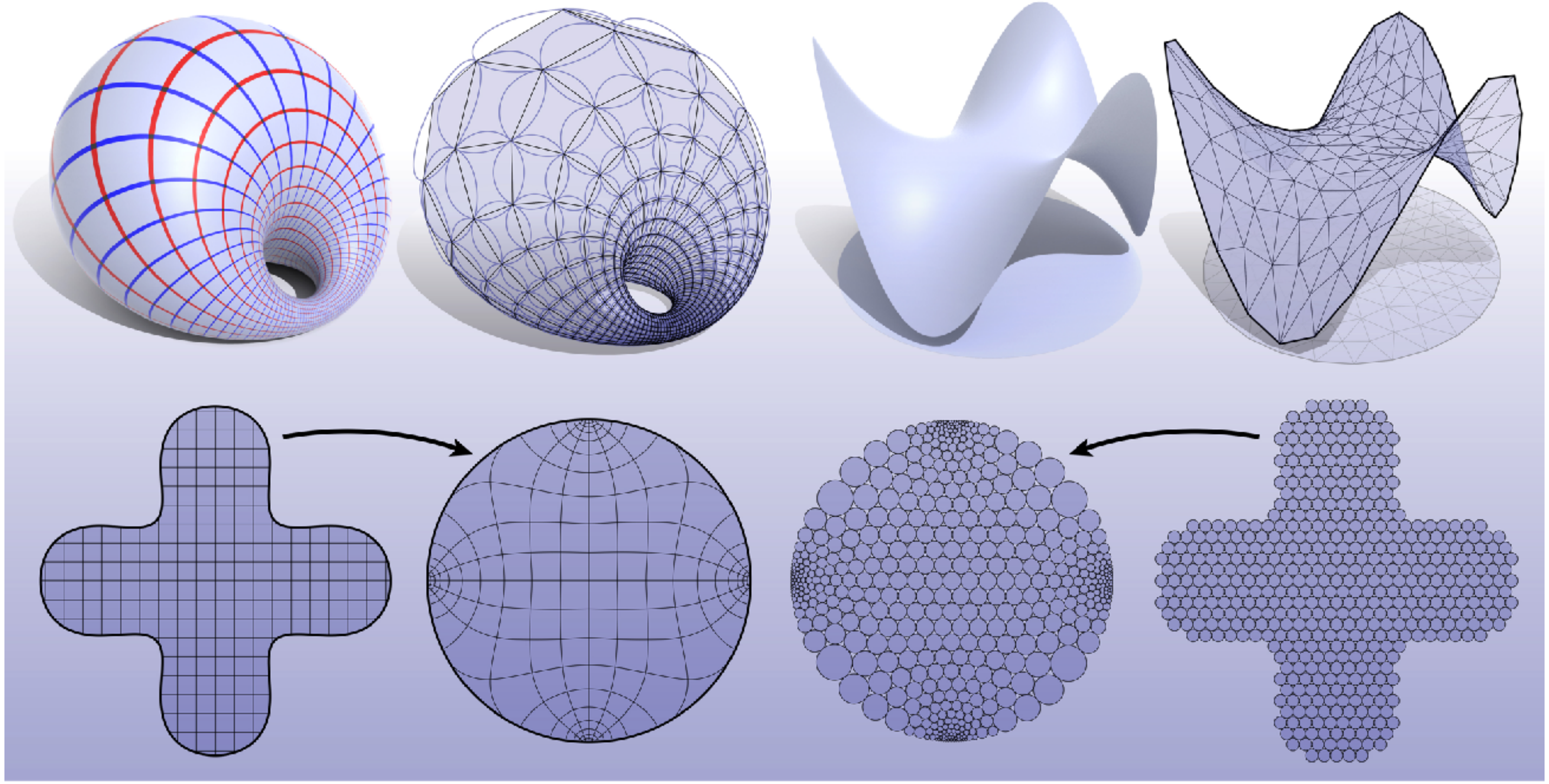
Acknowledgements

- Many thanks to those who made this course possible:
  - Tom Barr, Lori Melucci, AMS Short Course Subcommittee
  - Joel Hass, Frank Morgan, AMS Notices Editorial Board
  - Joint Mathematics Meetings 2018 (MAA / AMS)
  - National Science Foundation (Award #1717320)

MAA MATHEMATICAL ASSOCIATION OF AMERICA AMS NSF

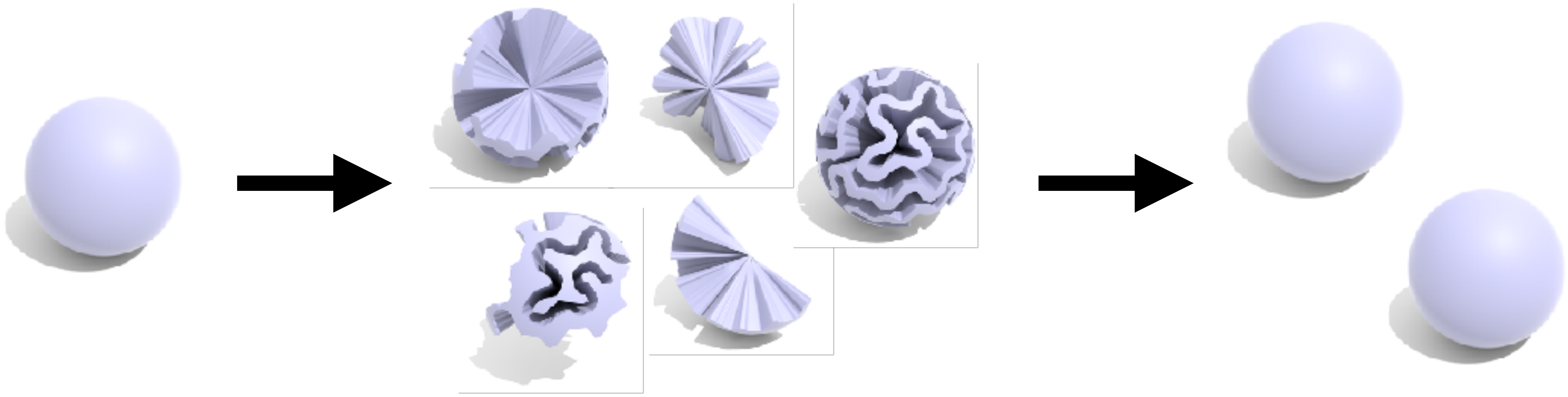
What Is Discrete Differential Geometry?

Basic idea: re-imagine ideas from classical differential geometry, *without reference to differential calculus.*



Geometry and Finitism

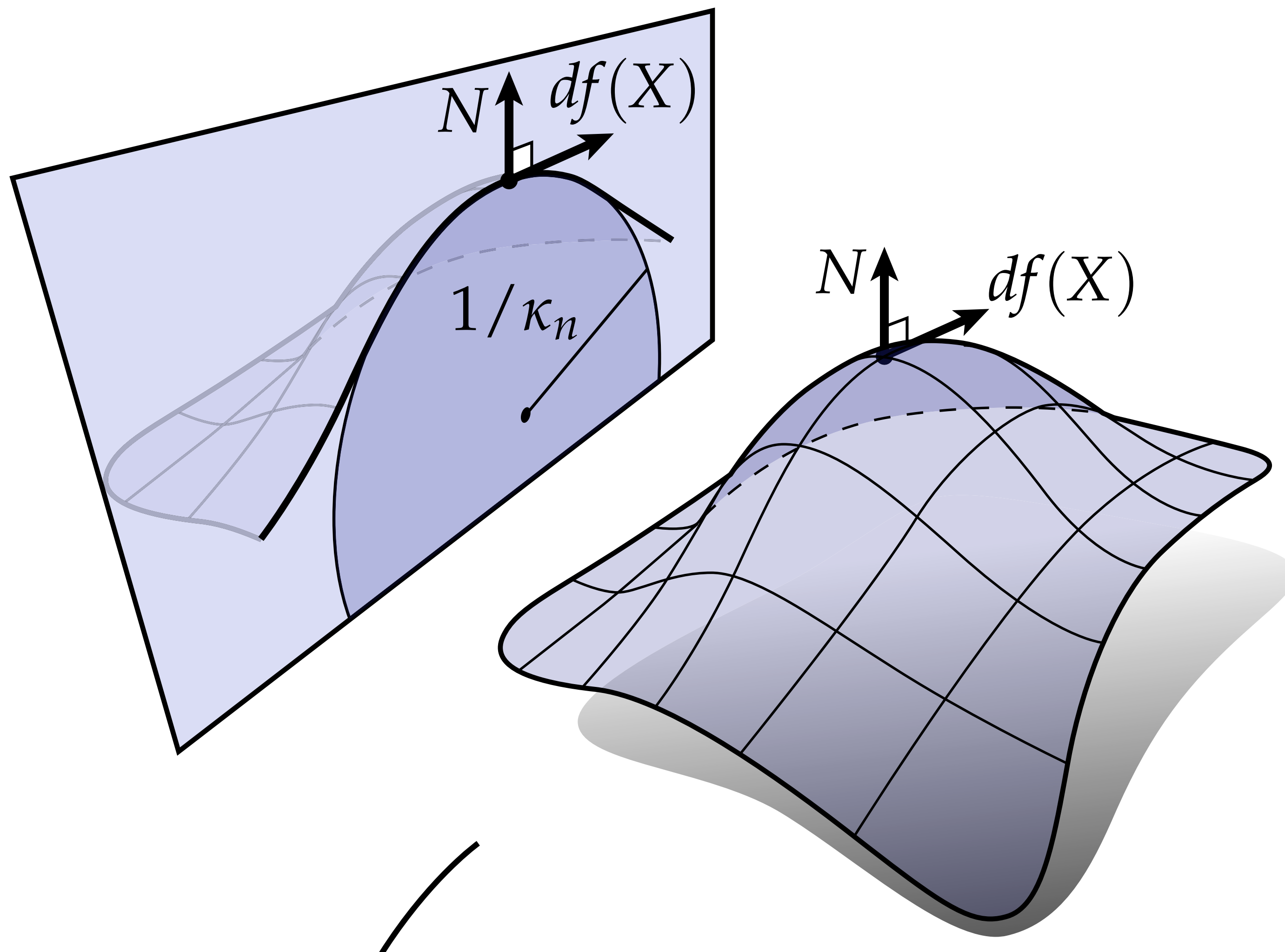
- Long and contentious history of *infinity* in mathematics*
  - Pythagorists viewed infinity as *evil!*
  - Later (~19th c.) learned to appreciate utility of infinity…
  - Can still have bizzare consequences (*e.g.*, Banach-Tarski)
  - **Finitists**: only "real" objects are those w/ finite descriptions
  - **Computer science**: finitism is a just practical matter…

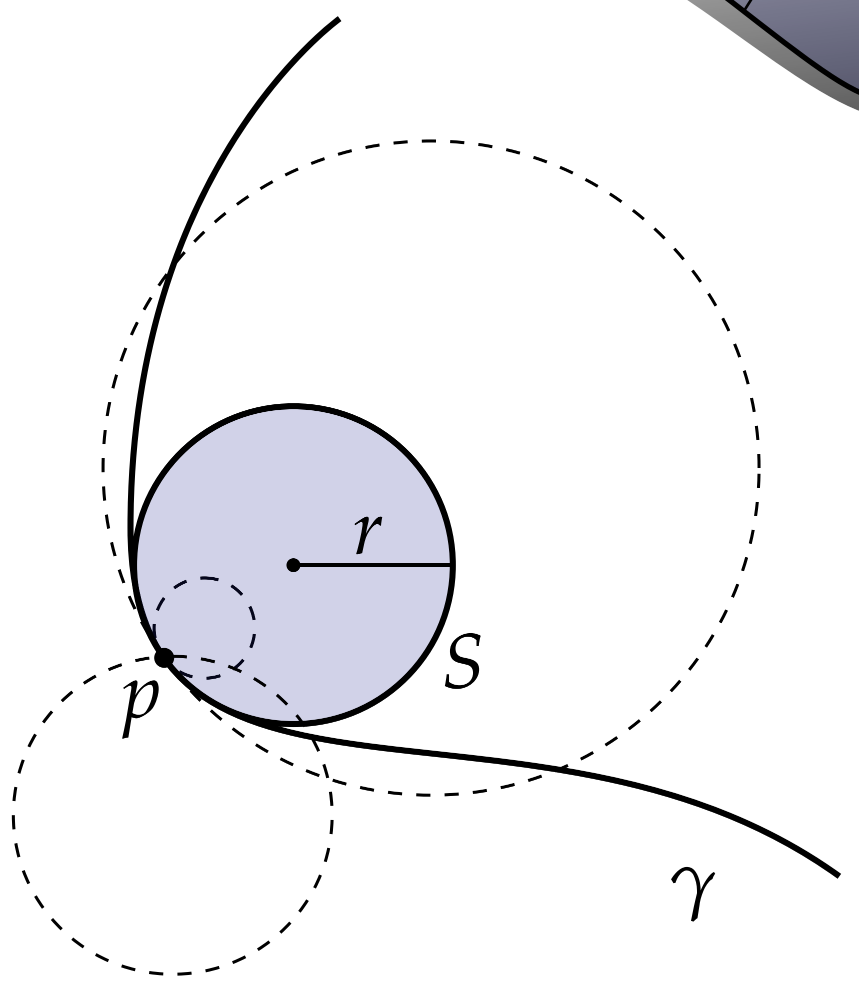


*Great series: *BBC Radio 4 — A History of the Infinite*

What is Differential Geometry?

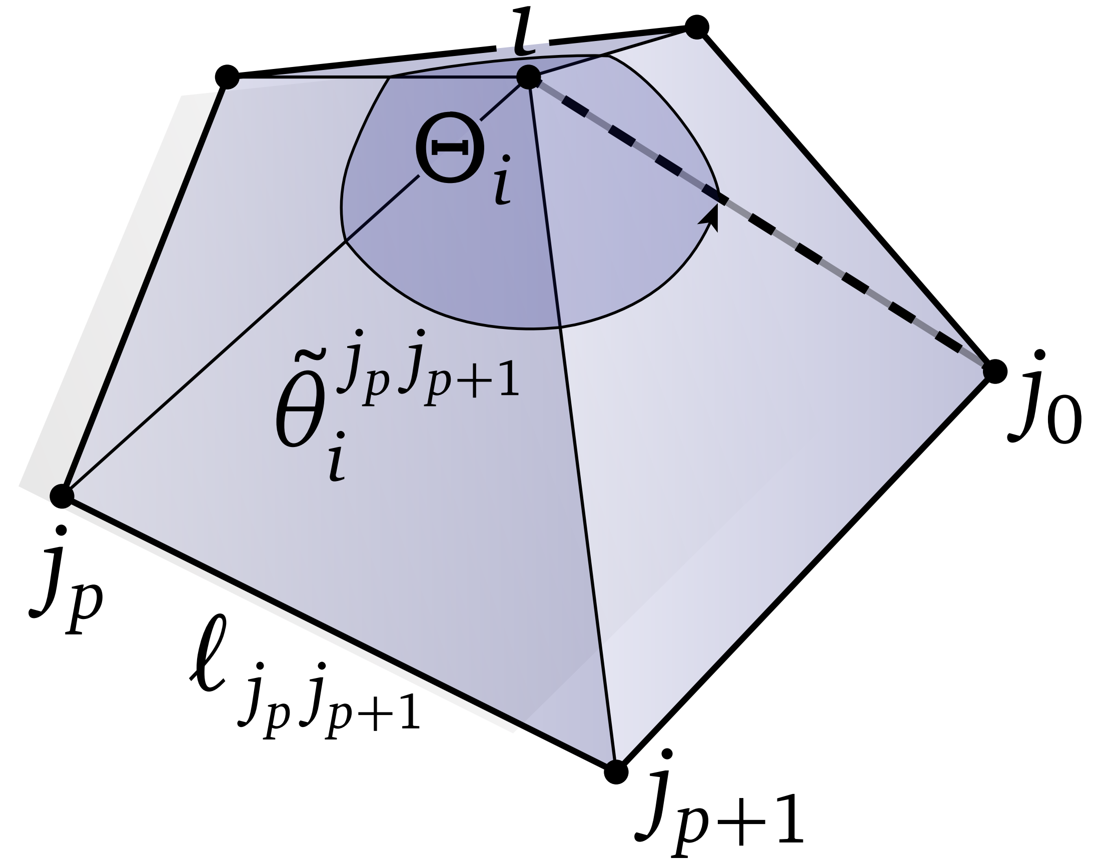
- **Language** for talking about *local properties of shape*
  - How fast are we traveling along a curve?
  - How much does the surface bend at a point?
  - etc.
- …and their connection to *global properties of shape*
  - So-called “local-global” relationships.
- Modern language of geometry, physics, statistics, …
- Profound impact on scientific & industrial development in 20th century



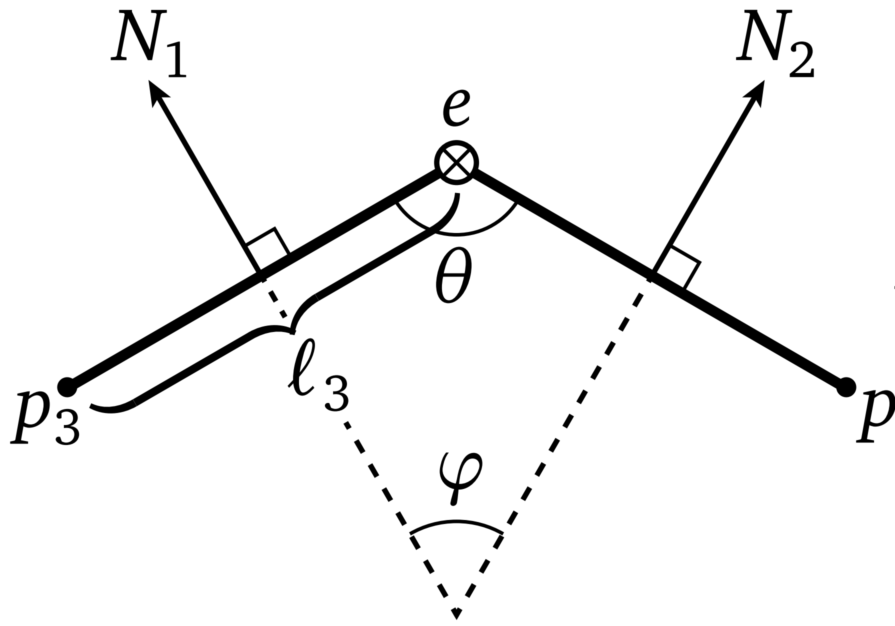


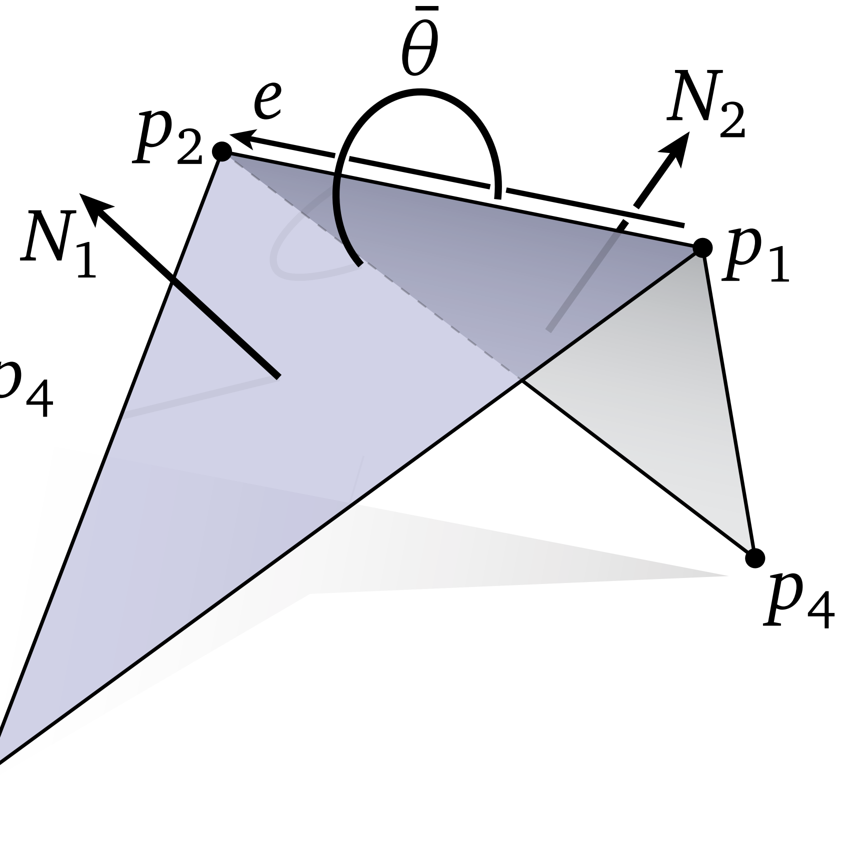
*What is **Discrete** Differential Geometry?*

- Also a language describing local properties of shape
  - *Infinity no longer allowed!*
  - No longer talk about derivatives, infinitesimals…
  - Everything expressed in terms of lengths, angles…

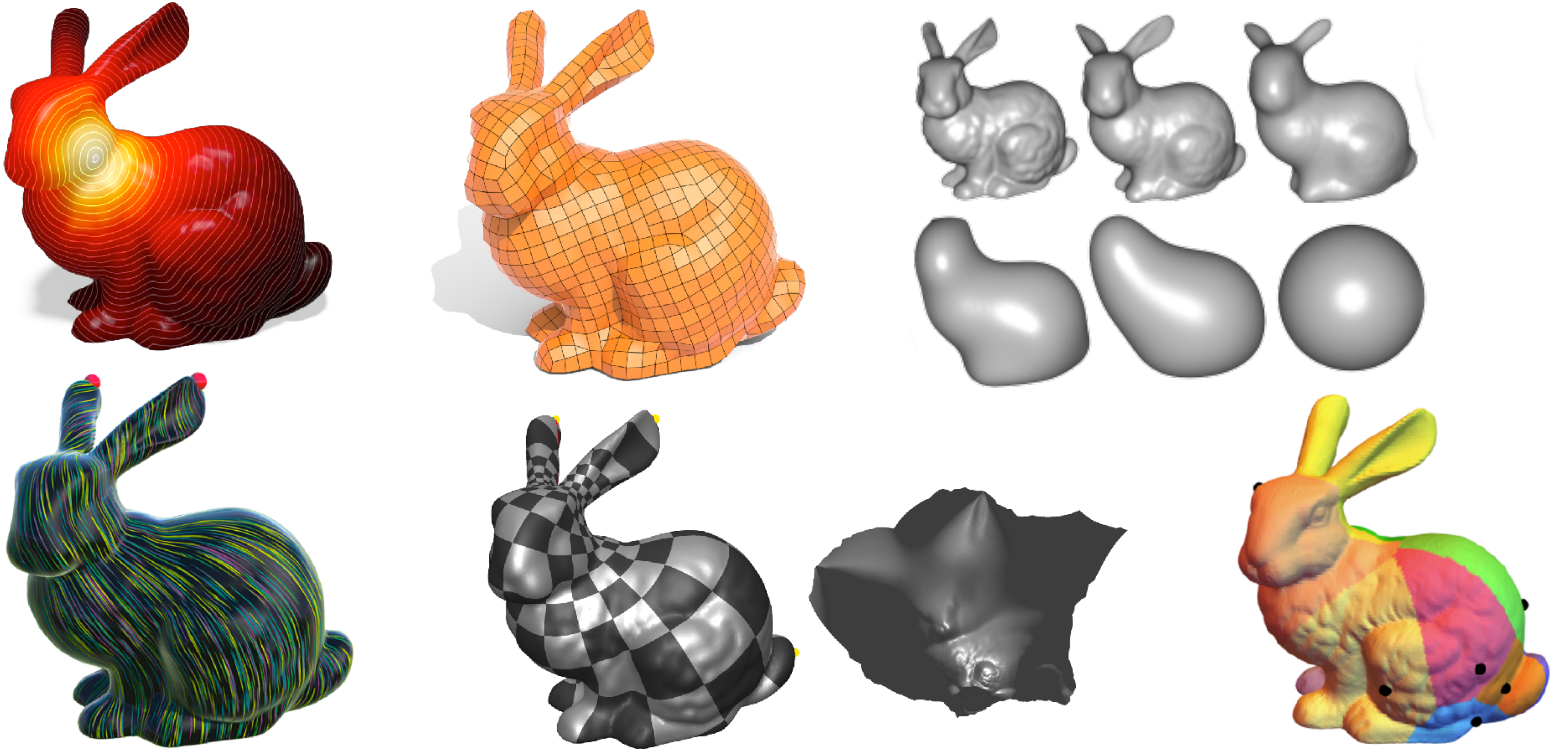


- Surprisingly little is lost!
  - Faithfully captures many fundamental ideas
- A modern language for geometric computing
- Increasing impact on science & technology in 21st century…

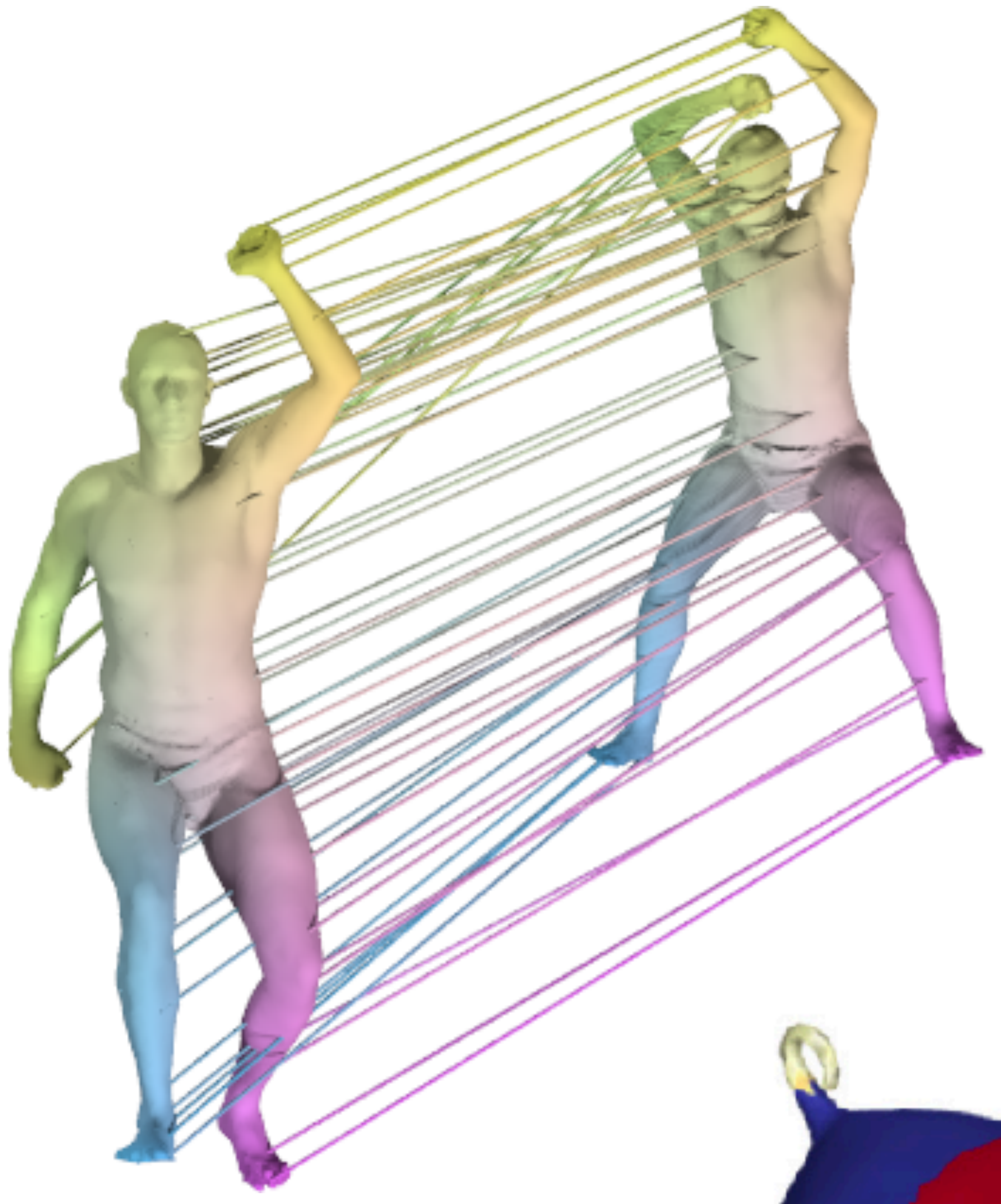


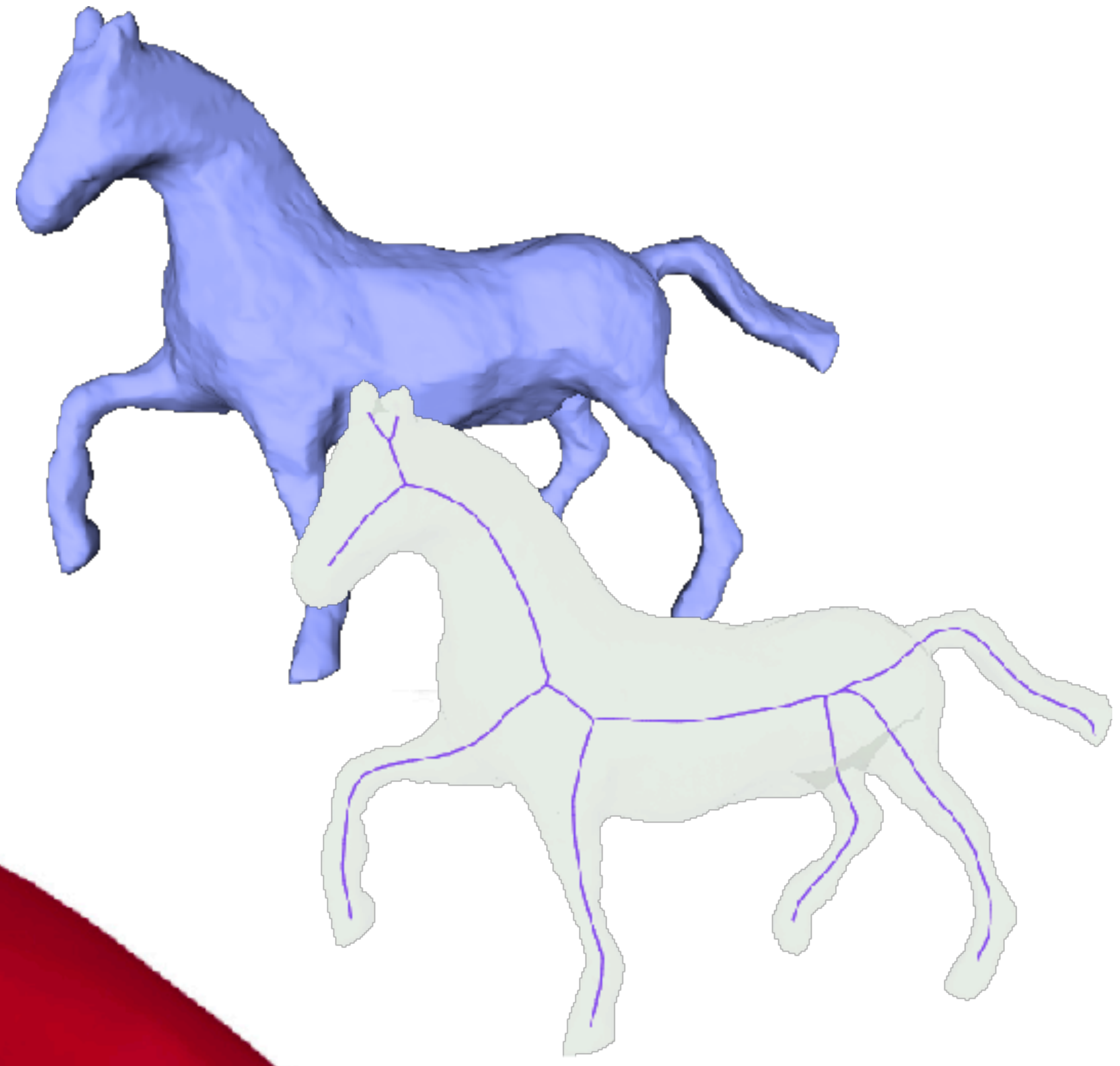


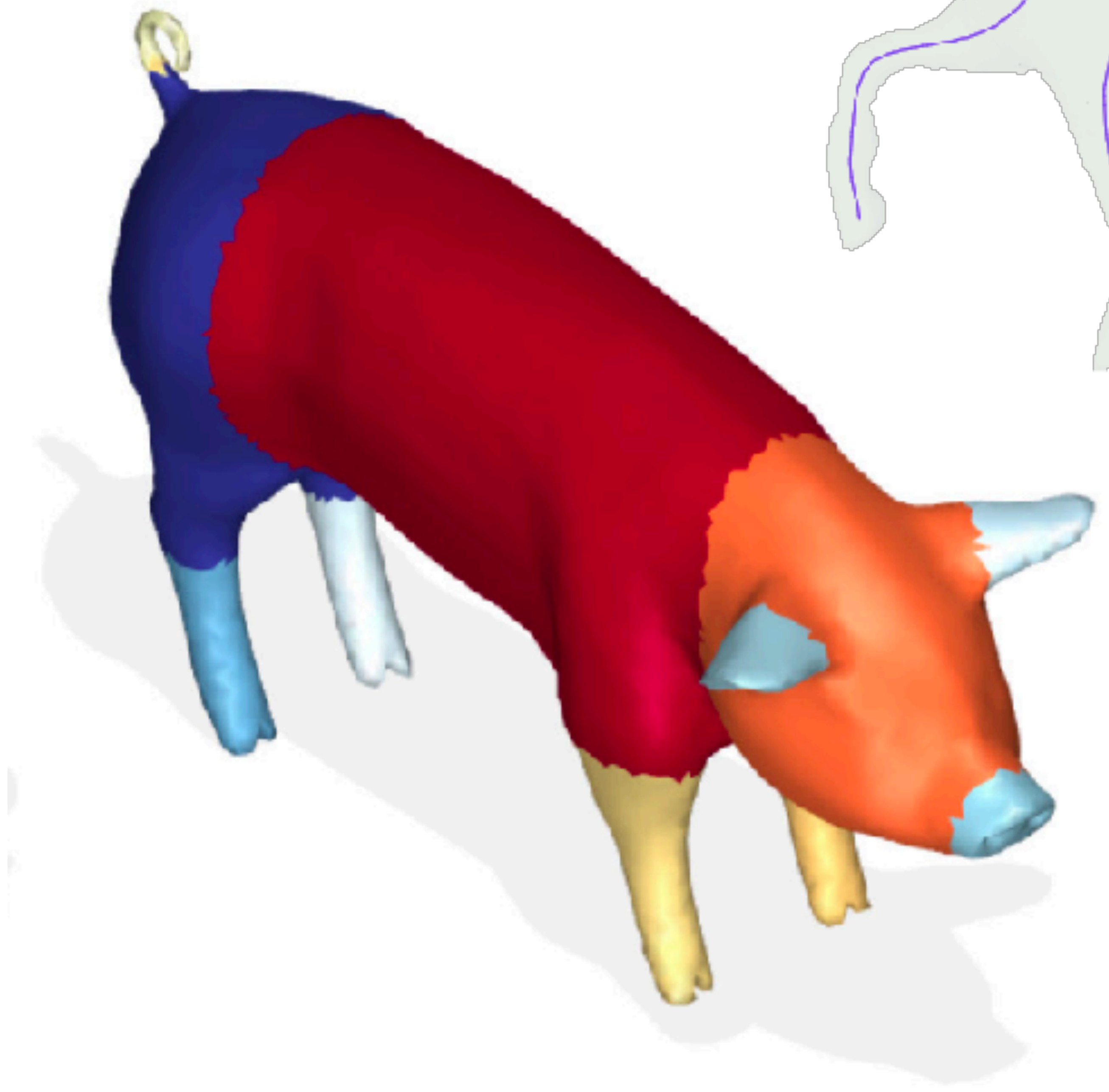
Applications of DDG: Geometry Processing

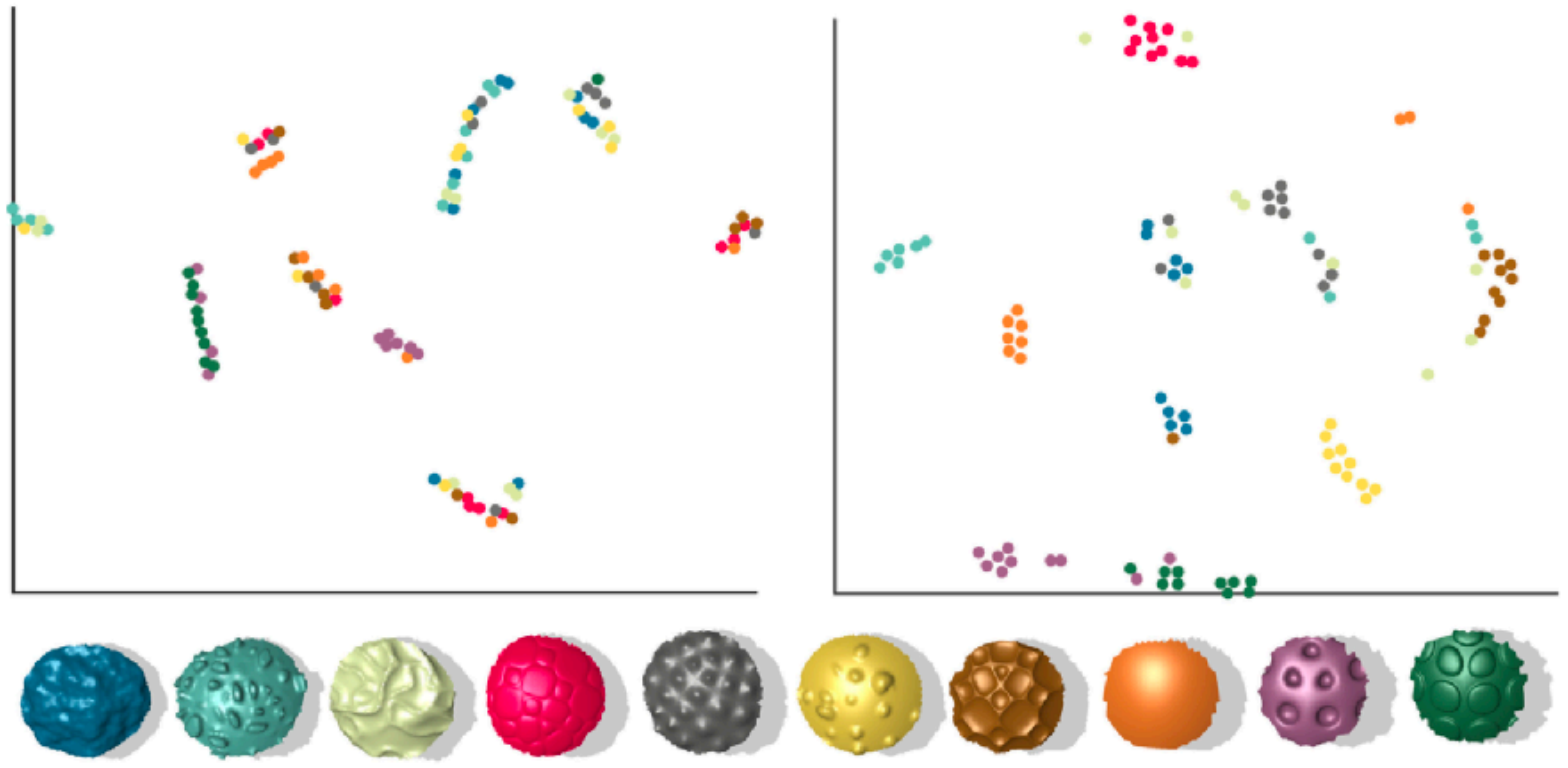


Applications of DDG: Shape Analysis





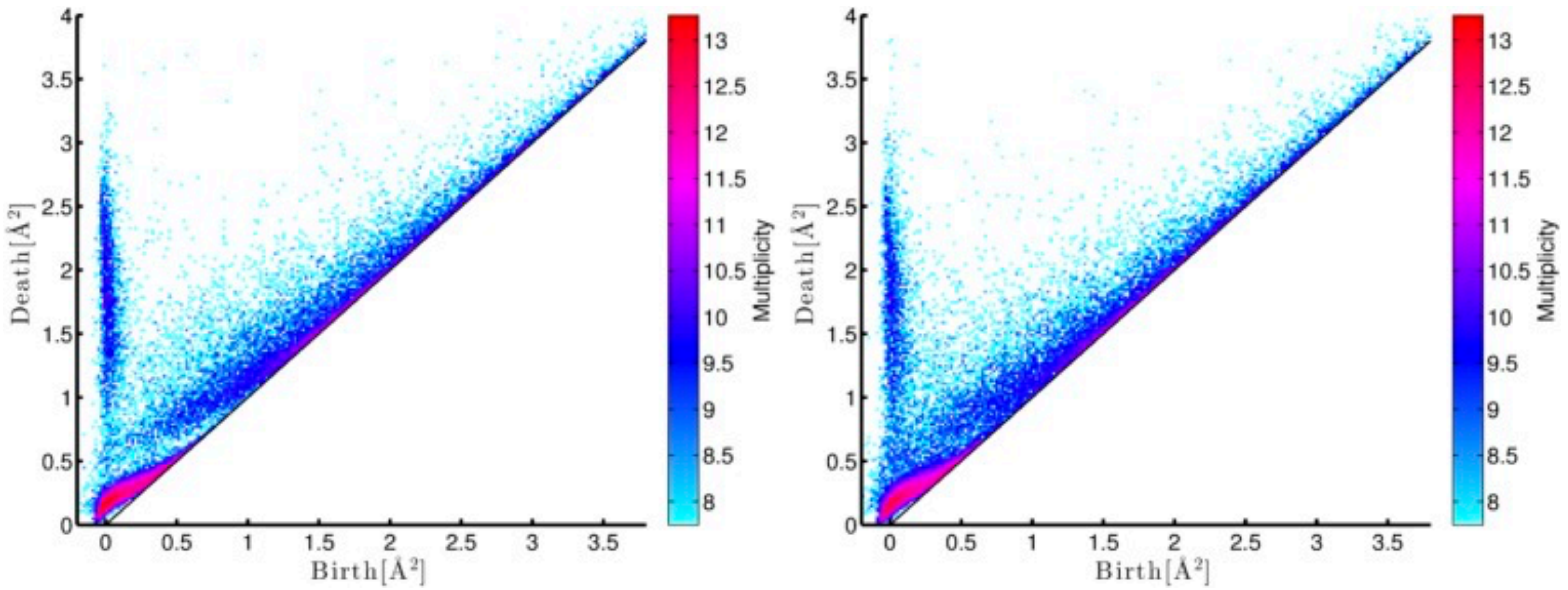


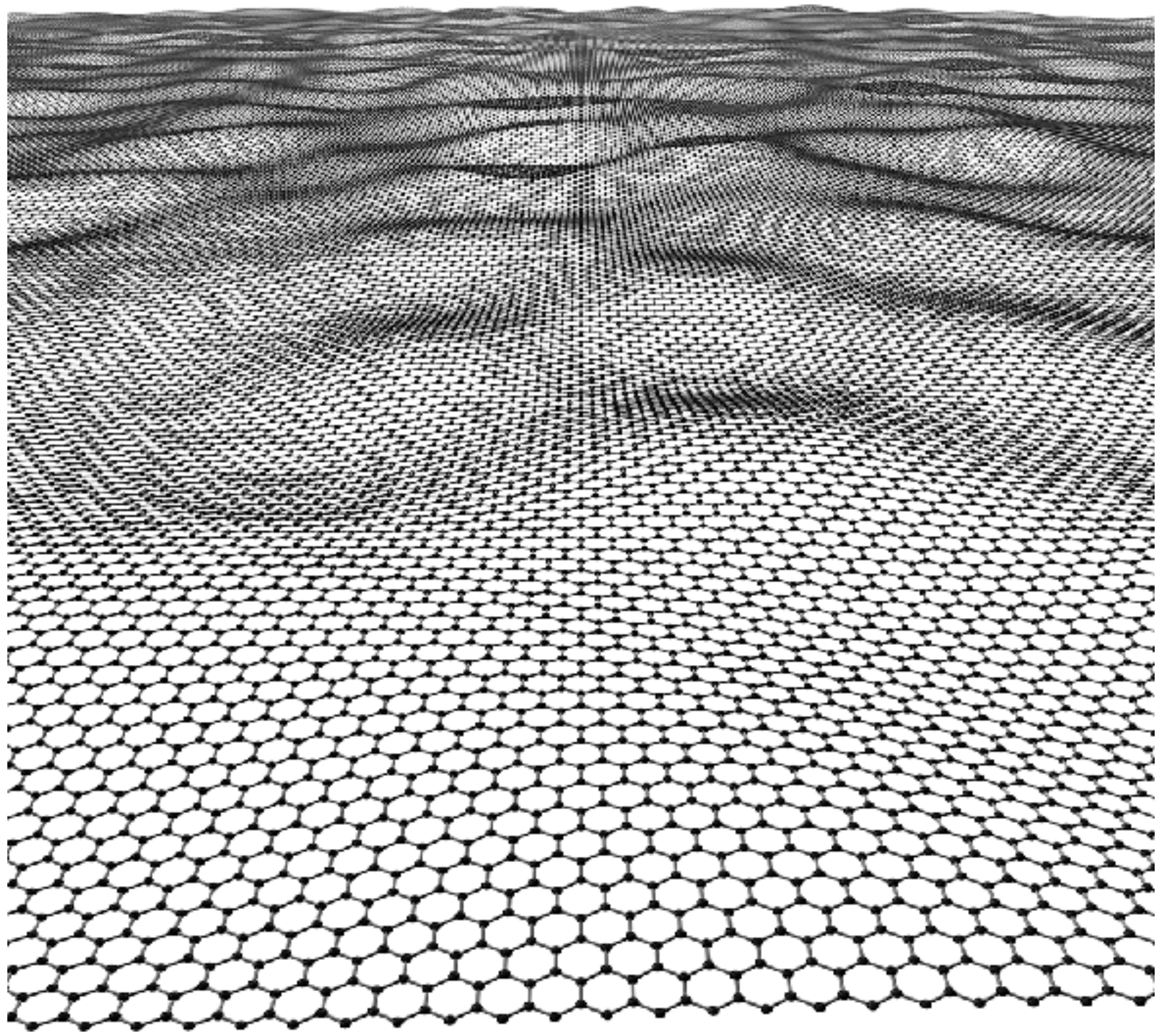


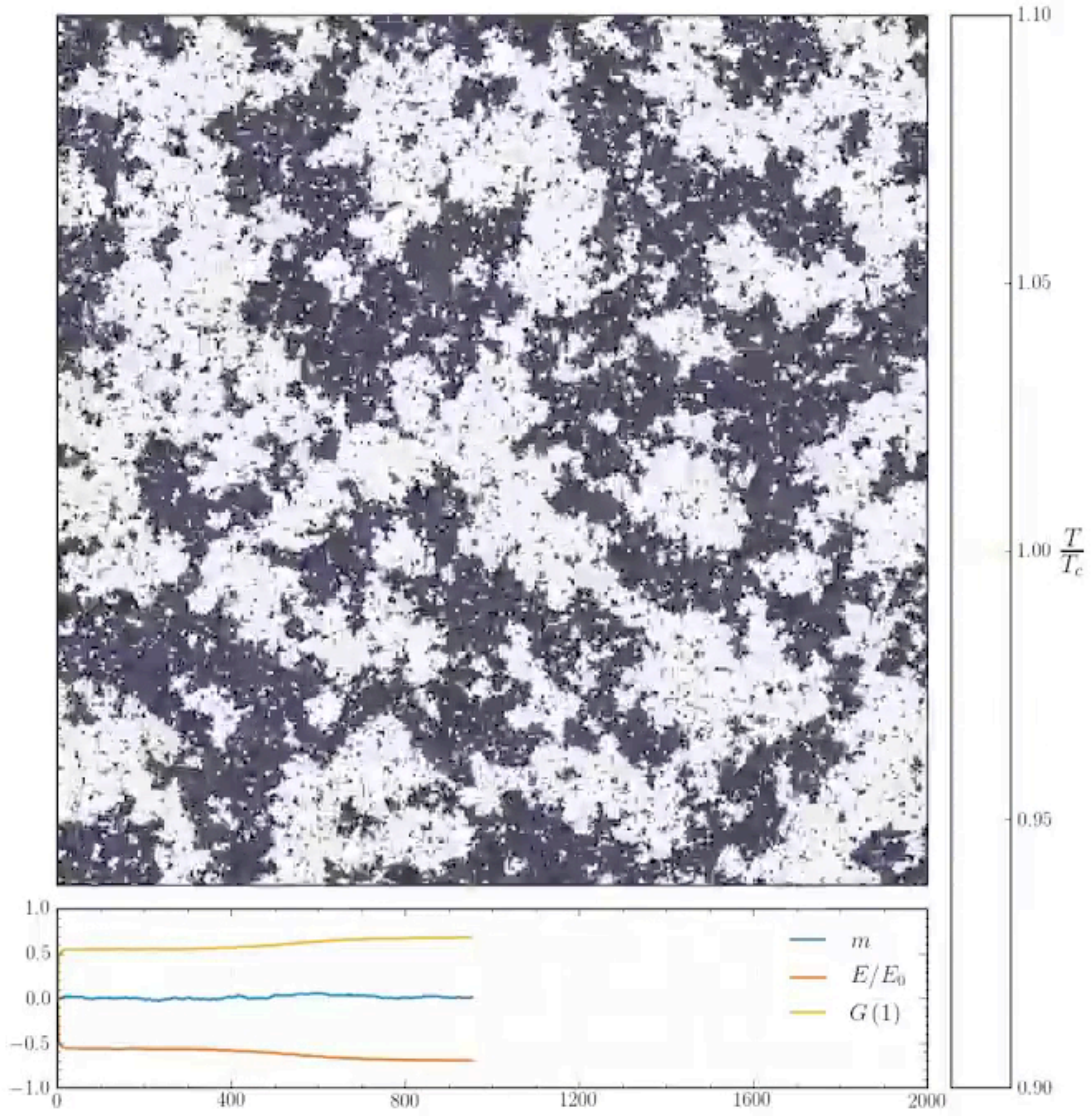
Applications of DDG: Numerical Simulation

Applications of DDG: Architecture & Design

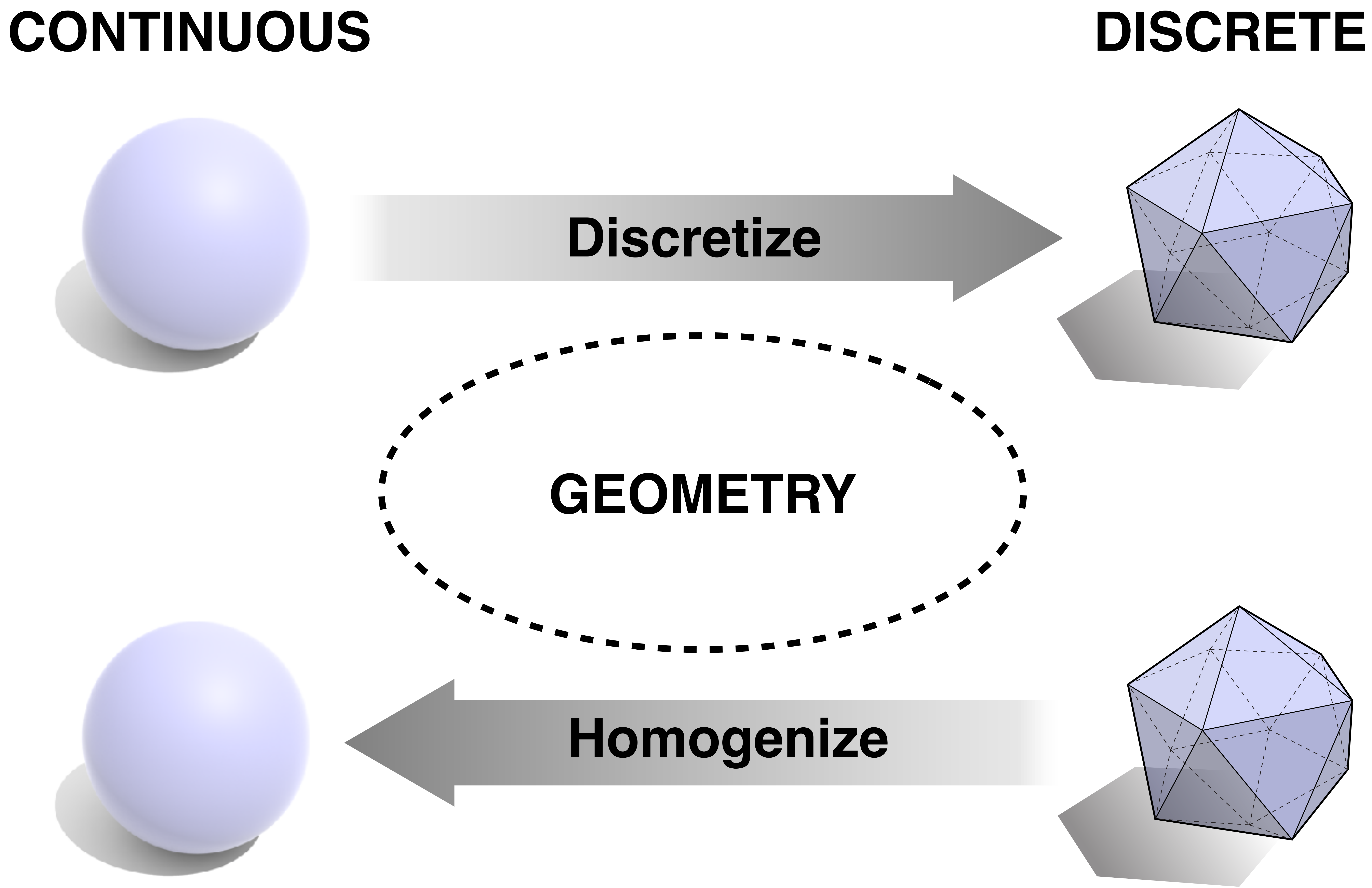
Applications of DDG: Discrete Models of Nature







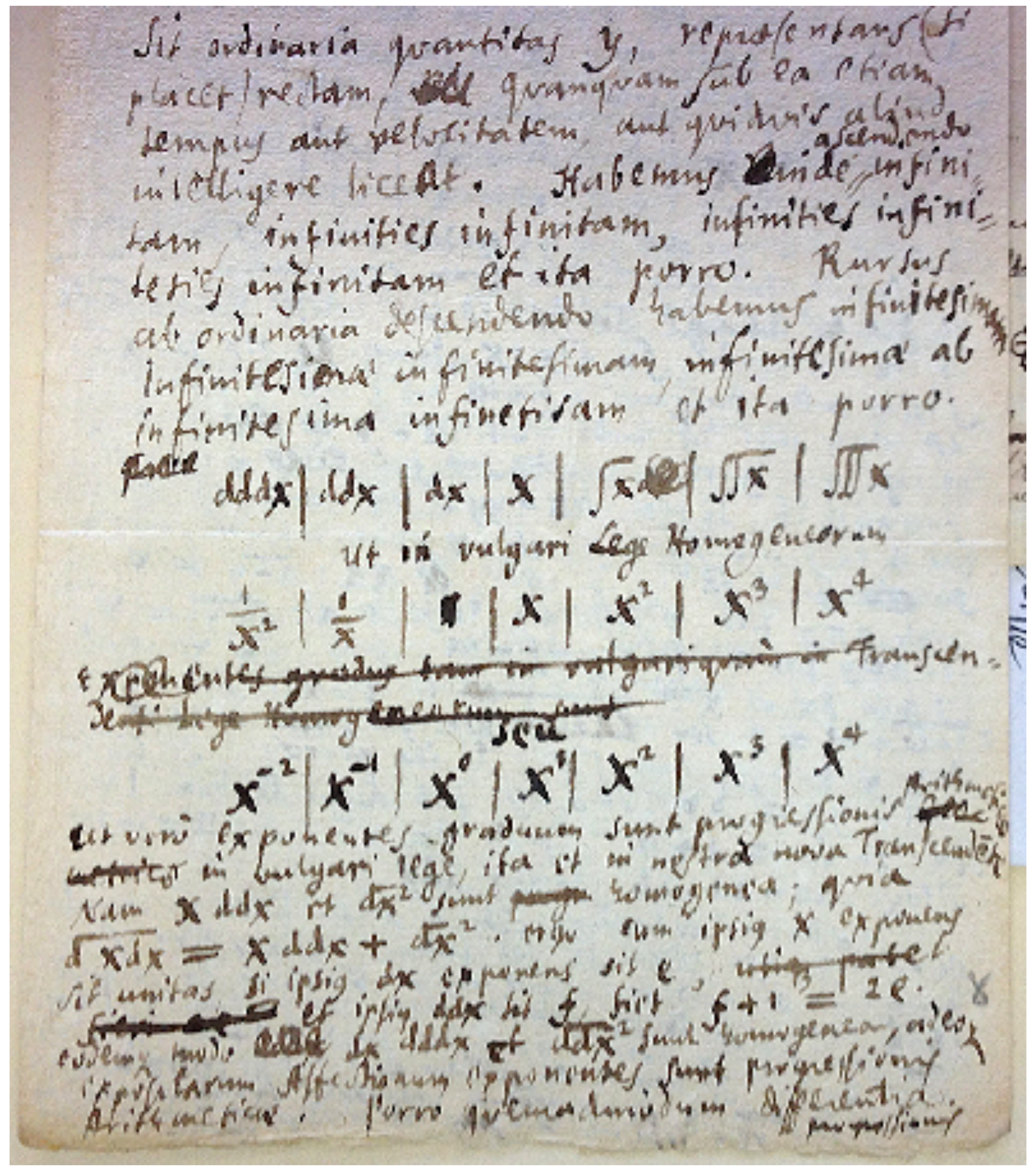
Discrete Differential Geometry

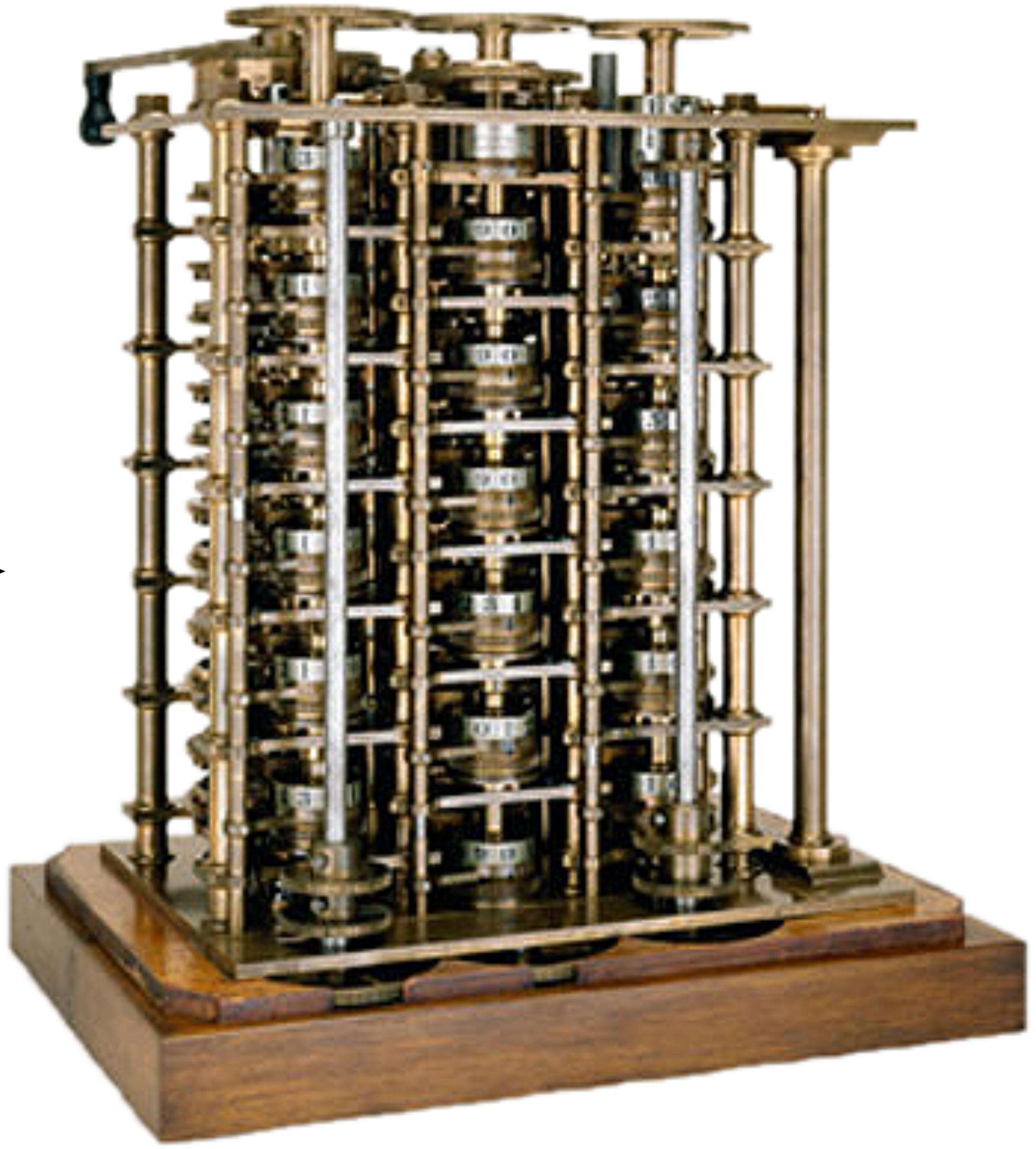


Discrete Differential Geometry—Grand Vision

GRAND VISION

Translate differential geometry into a **language** suitable for *computation*.

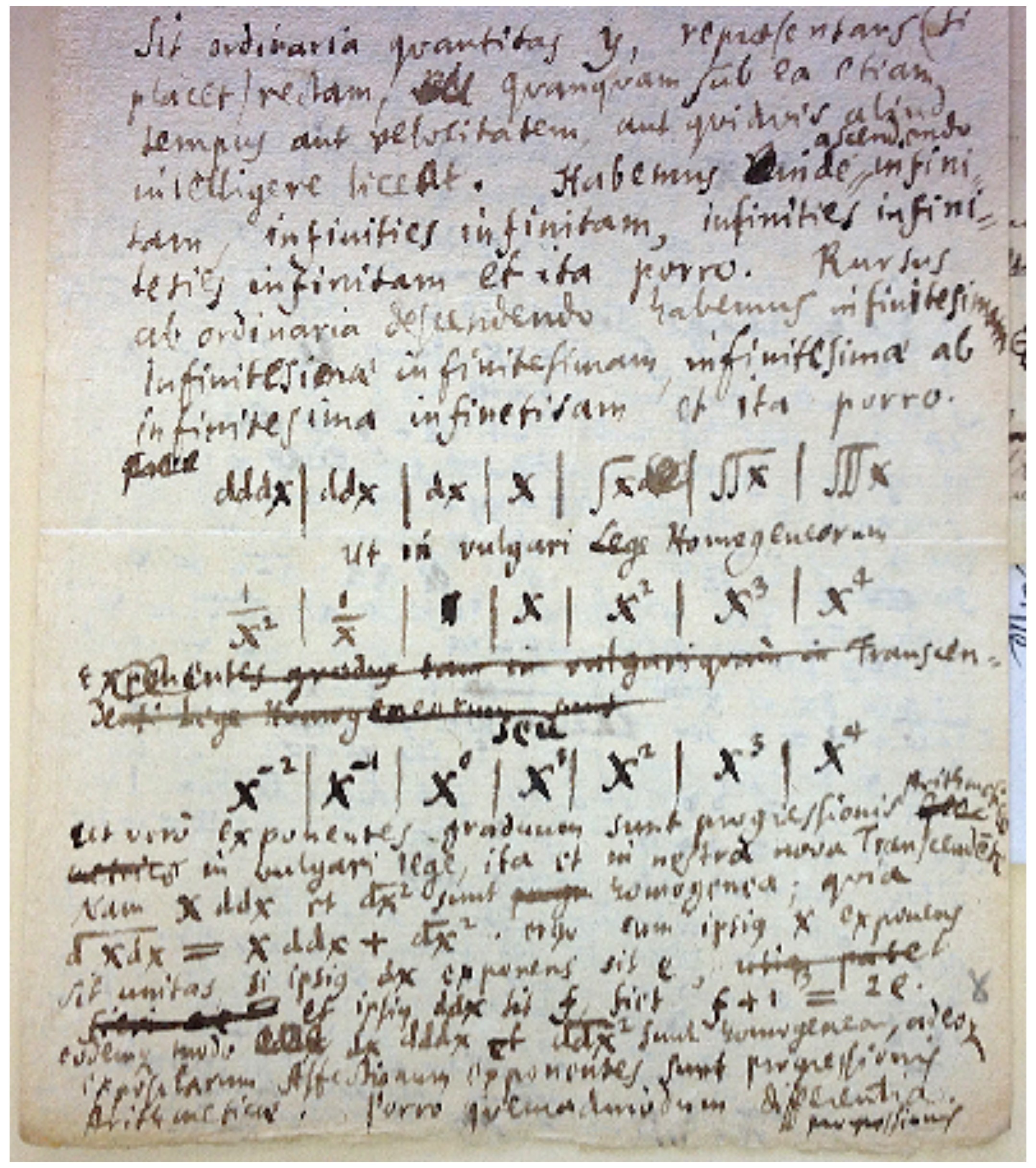


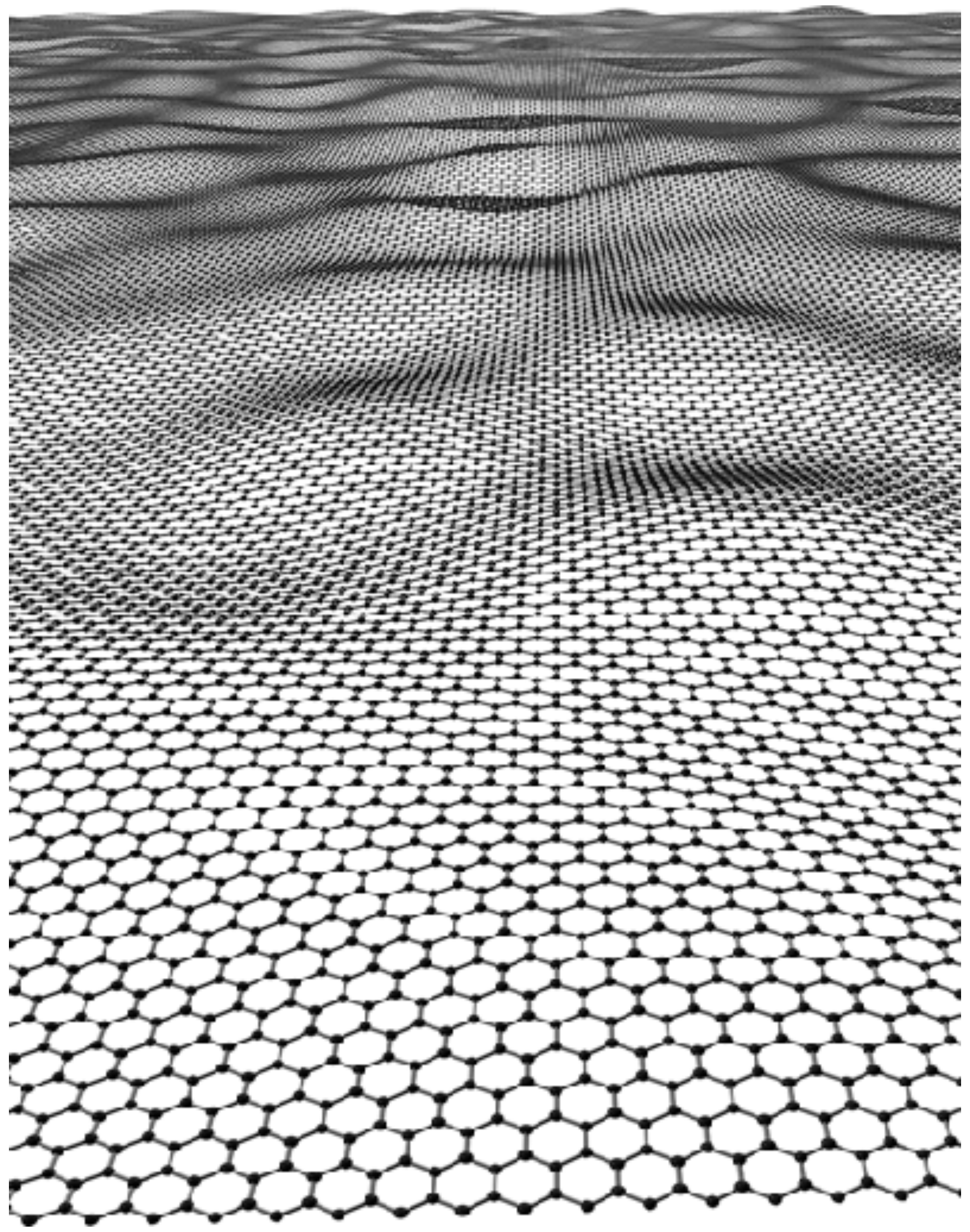


Discrete Differential Geometry—Grand Vision

GRAND VISION

Translate differential geometry into a **language** suitable for *modeling discrete phenomena*.





How can we get there?

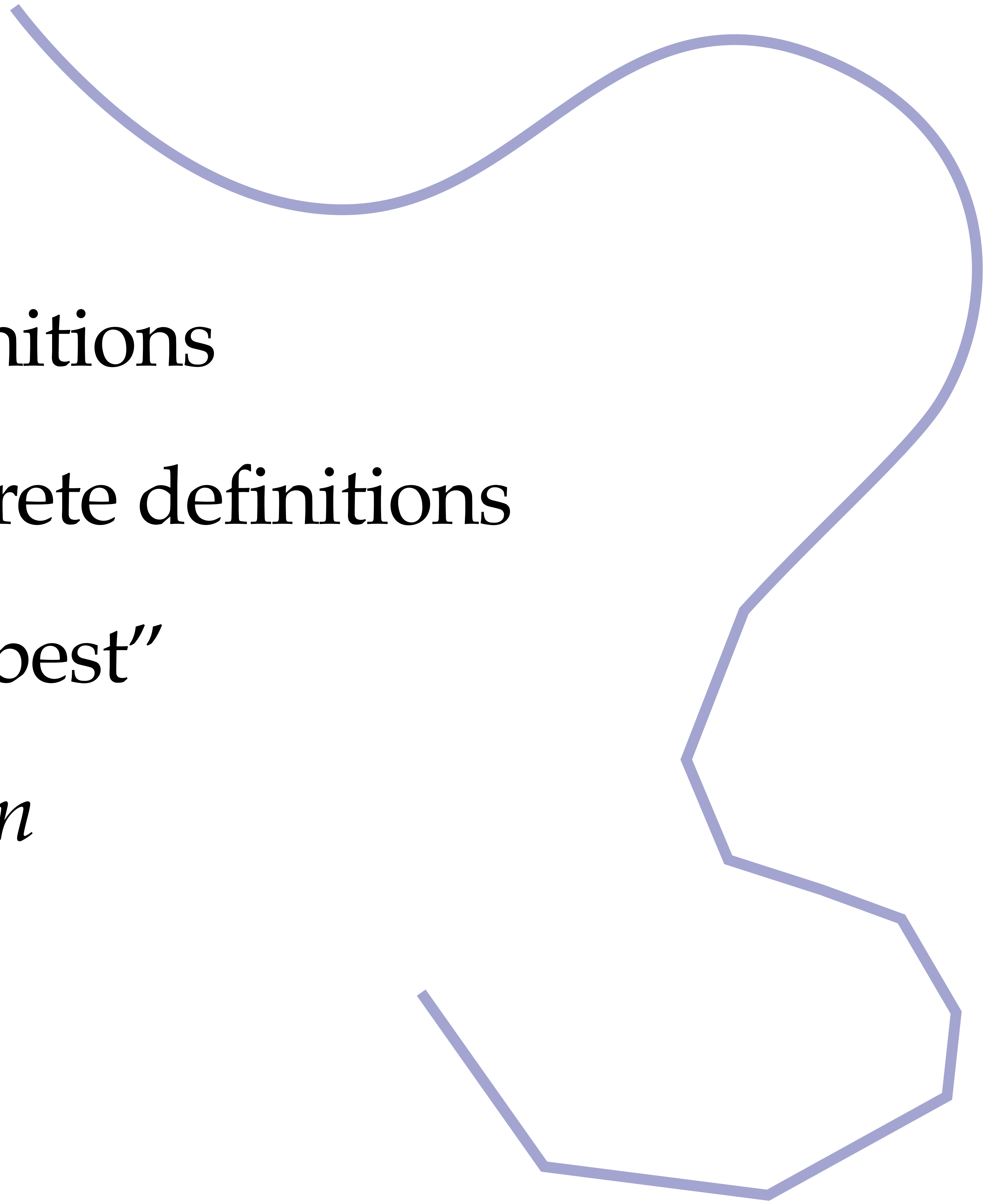
A common "game" is played in DDG to obtain discrete definitions:

1. Write down several **equivalent** definitions in the smooth setting.
2. Apply each smooth definition to an object in the discrete setting.
3. Determine which properties are captured by each resulting **inequivalent** discrete definition.

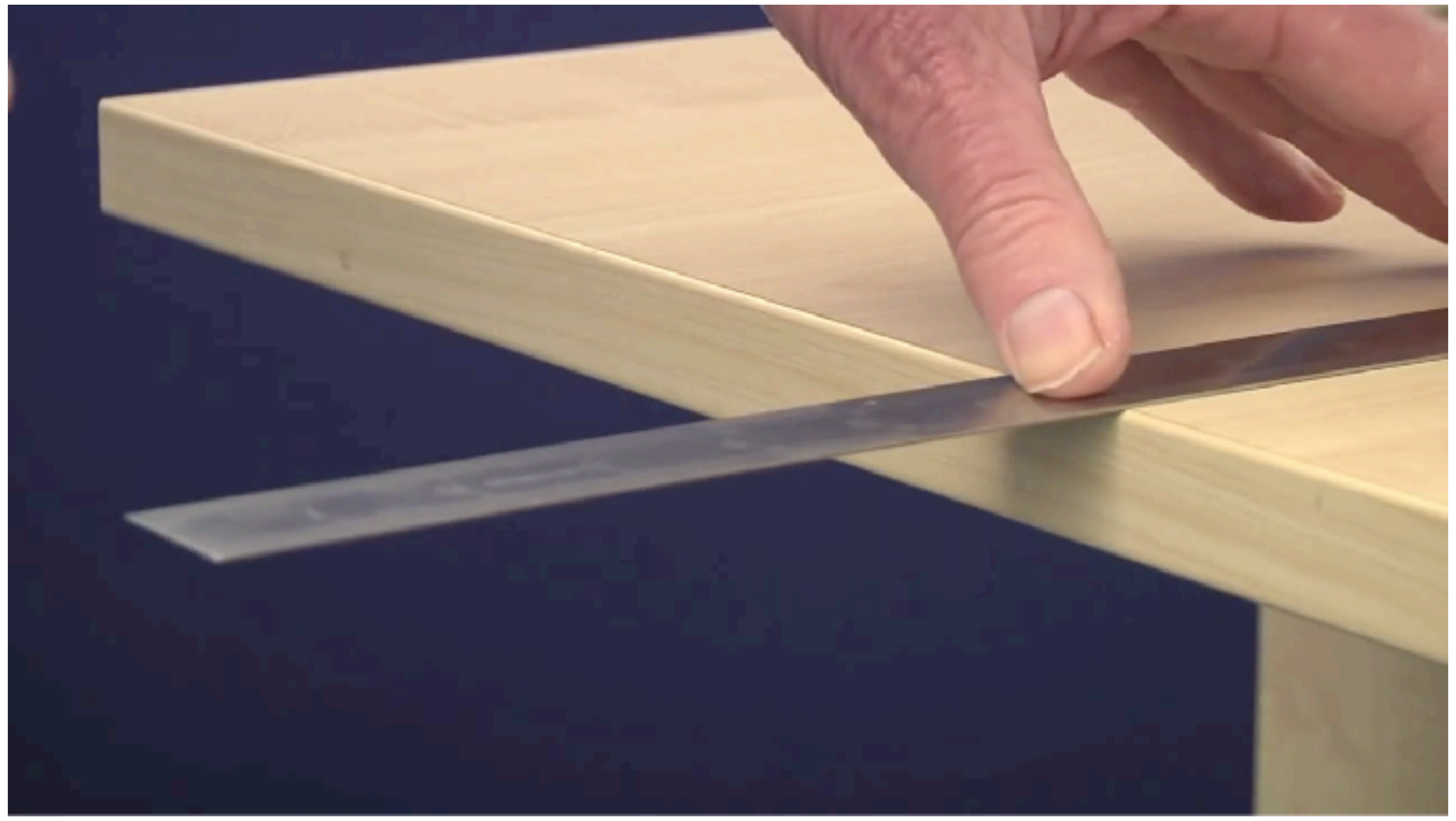
One often encounters a so-called *"no free lunch"* scenario: no single discrete definition captures *all* properties of its smooth counterpart.

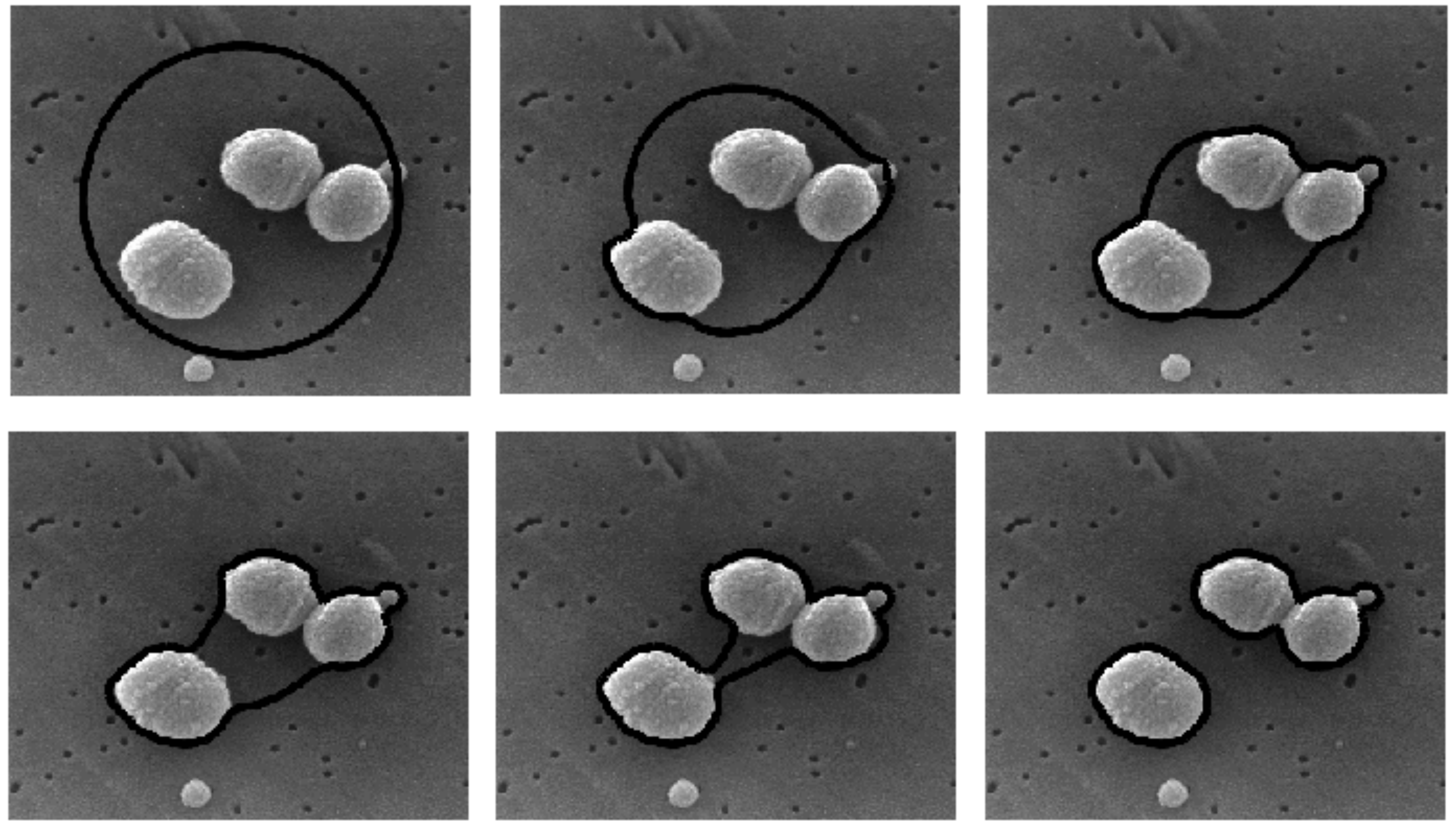
Example: Discrete Curvature of Plane Curves

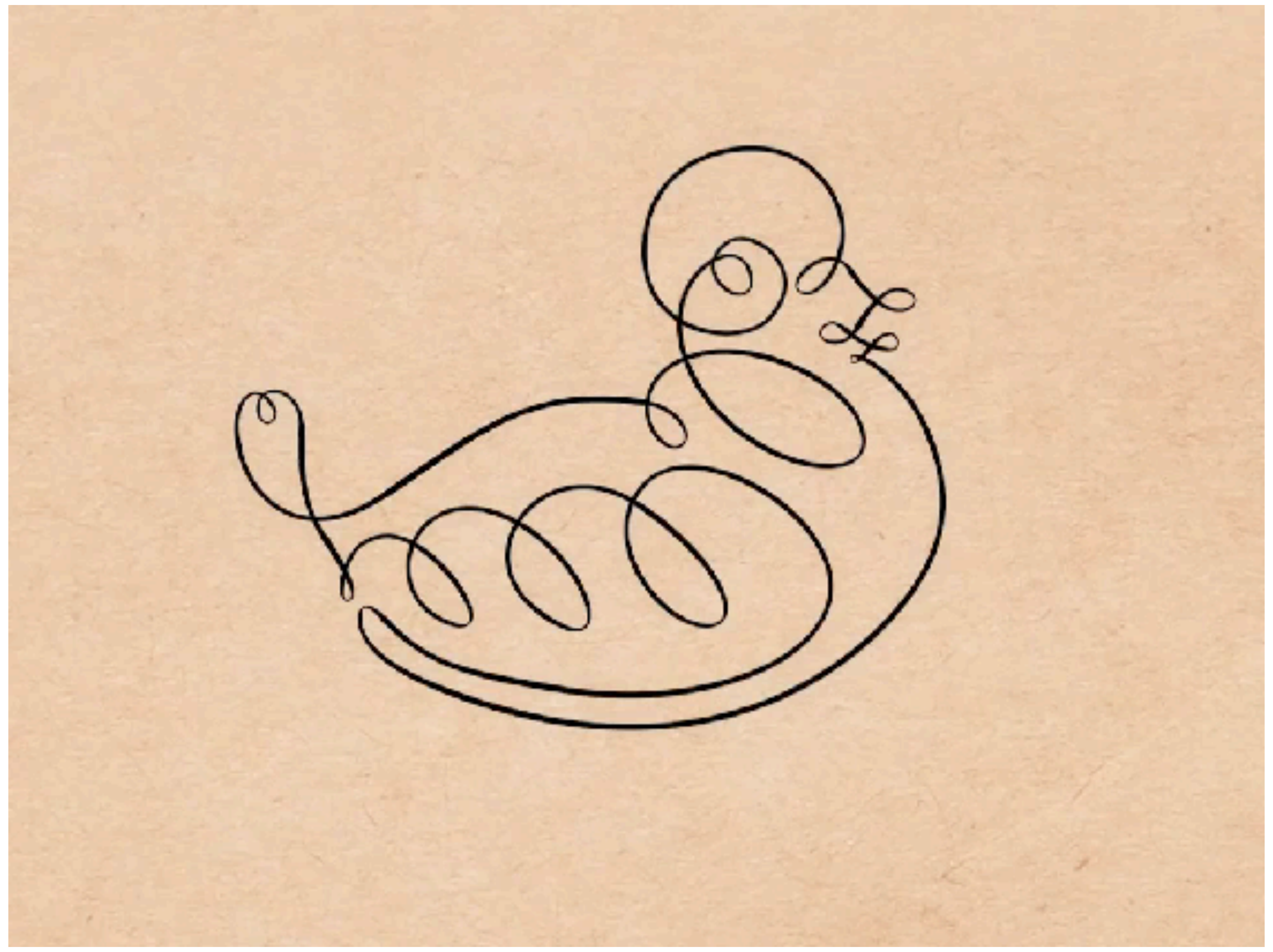
- **Toy example:** *curvature of plane curves*
  - Roughly speaking: “how much it bends”
  - First give several **equivalent** smooth definitions
  - Then play The Game to get **different** discrete definitions
  - Will discover that no single definition is “best”
  - *Pick the definition best suited to the application*
- **Very** brief overview
- Covered in more detail in Notices article



Curvature of a Curve — Motivation

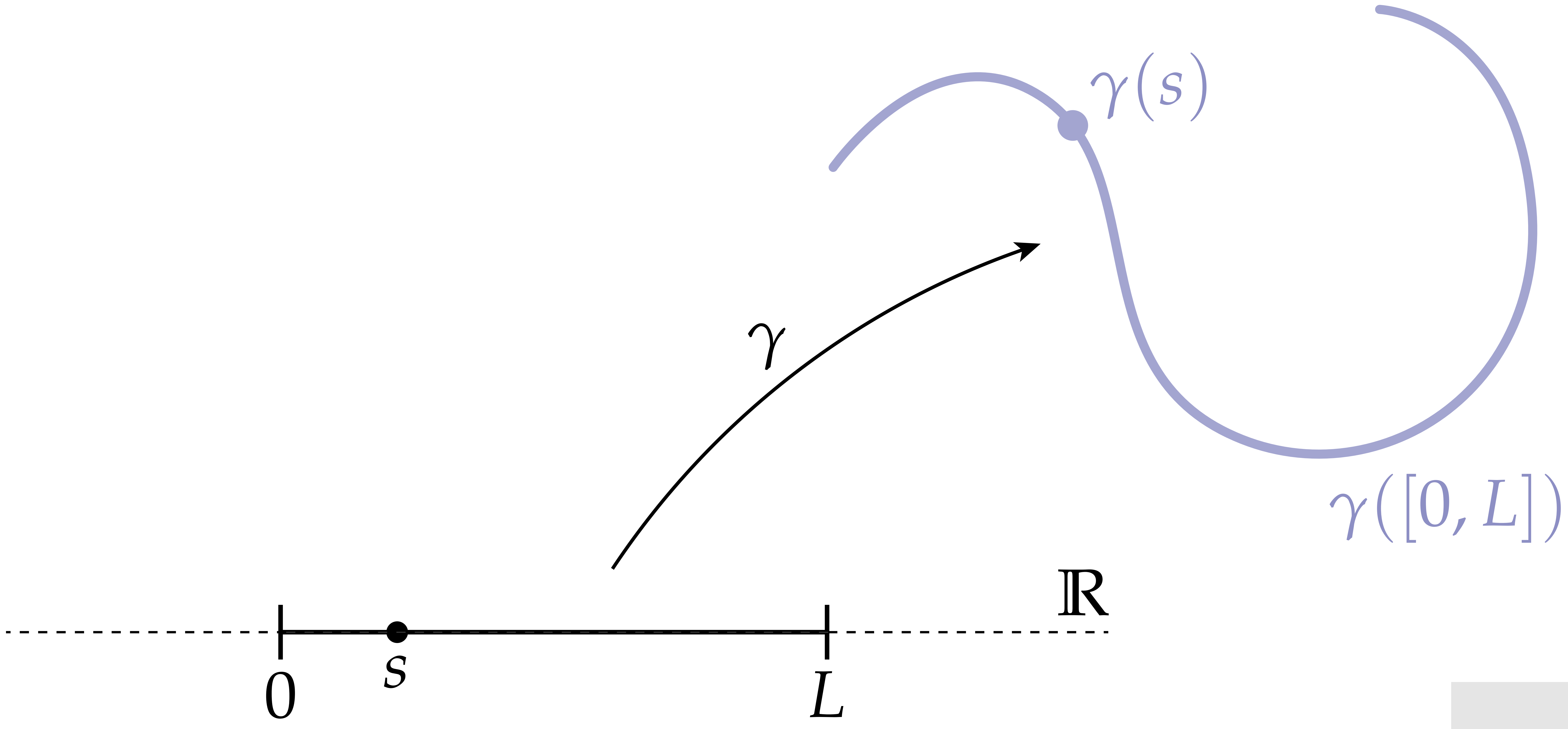






Curves in the Plane

- In the smooth setting, a **parameterized curve** is a map* taking each point in an interval $[0,L]$ of the real line to some point in the plane $\mathbb{R}^2$:



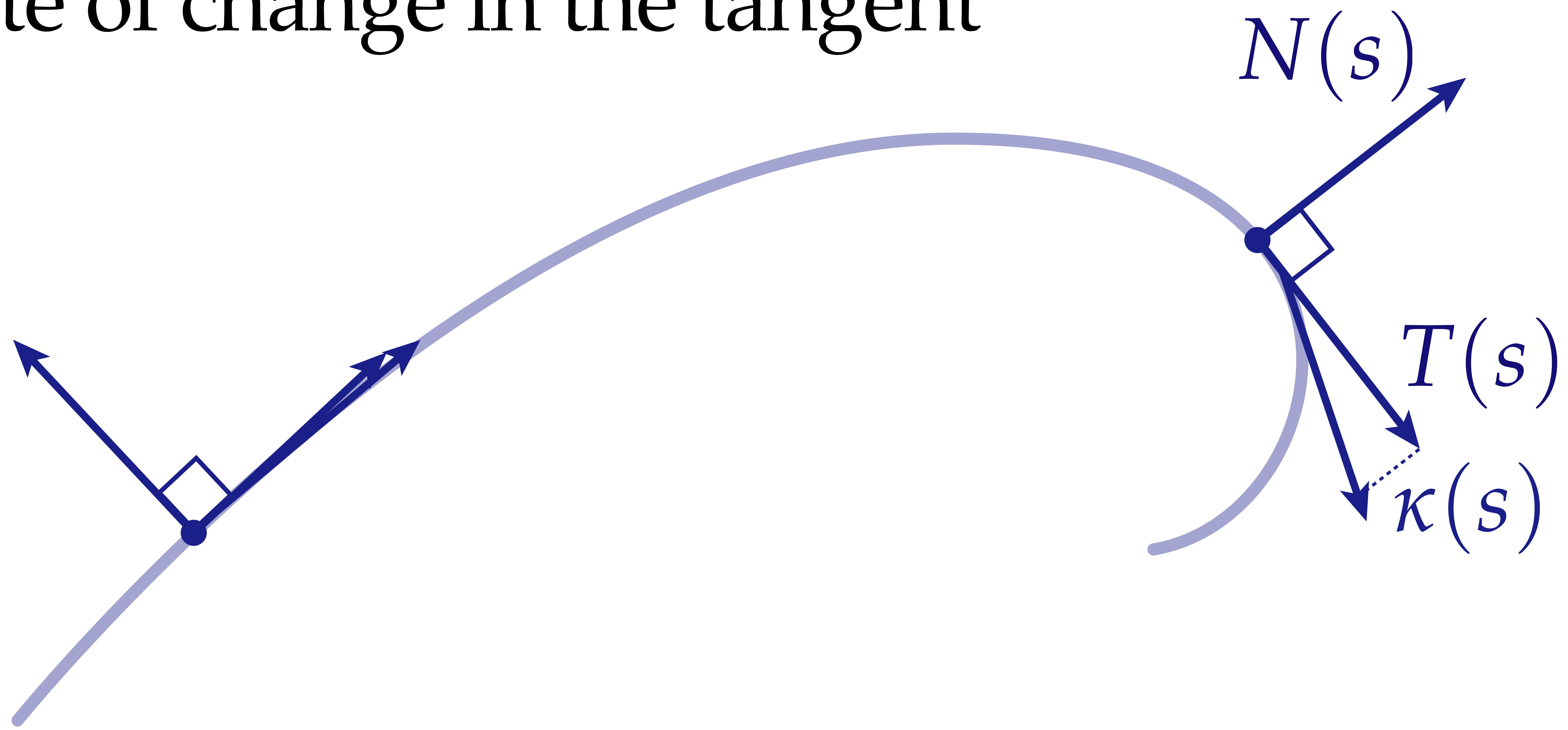
*Continuous, differentiable, smooth…

$$\gamma : [0,L] \to \mathbb{R}^2$$

Curvature of a Smooth Curve

- Informally, curvature describes "how much a curve bends"
- More formally, the **curvature** of an arc-length parameterized plane curve can be expressed as the rate of change in the tangent*

$$\kappa(s) := \langle N(s), \tfrac{d}{ds} T(s) \rangle$$
$$= \langle N(s), \tfrac{d^2}{ds^2} \gamma(s) \rangle$$



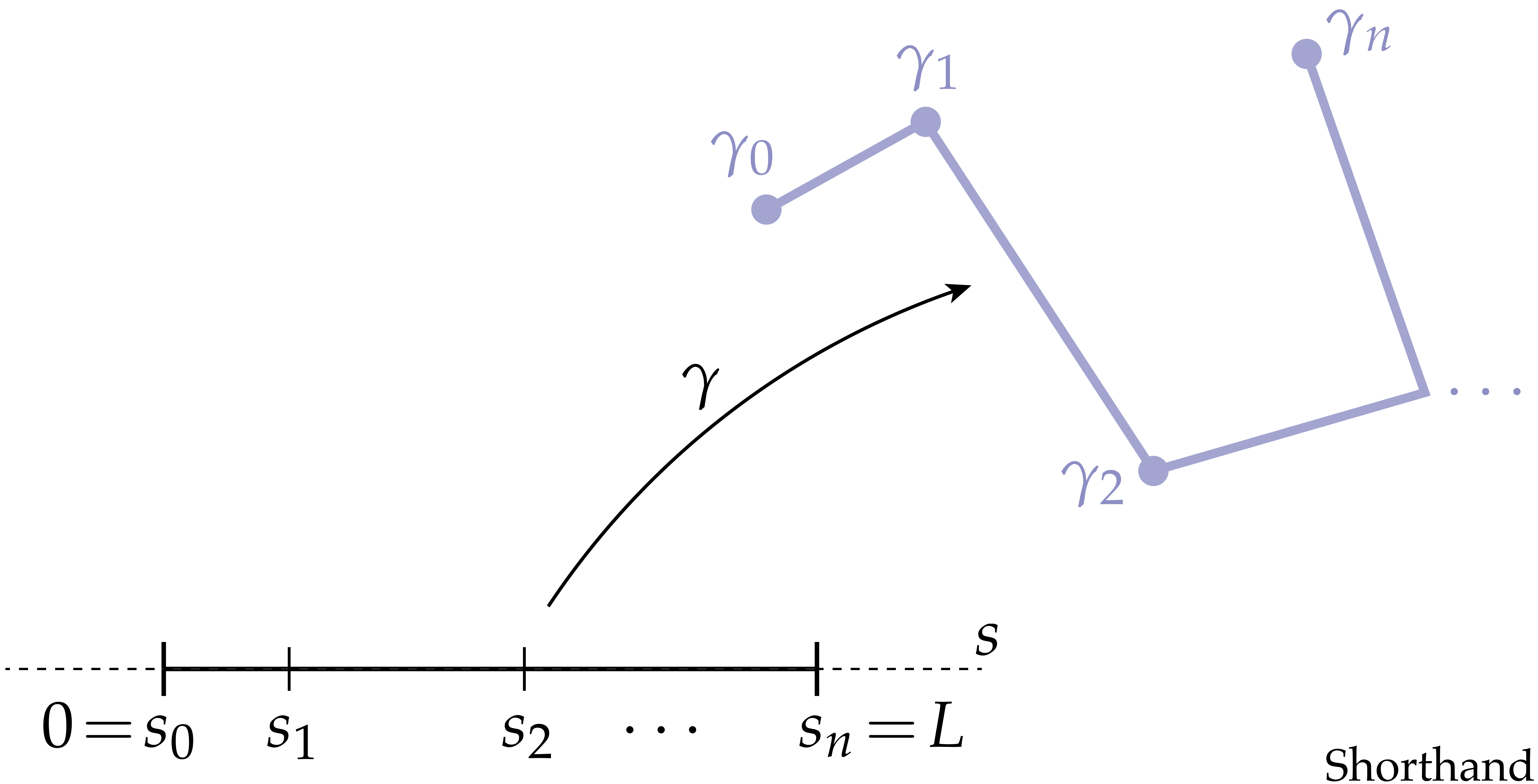
KEY IDEA

Curvature is a *second derivative*.

*Here the angle brackets denote the usual dot product, i.e., $\langle (a,b),(x,y) \rangle := ax + by$.

Discrete Curves in the Plane

- A **discrete curve** is a *piecewise linear* parameterized curve, *i.e.*, a sequence of **vertices** connected by straight line segments:



Shorthand: $\gamma_i := \gamma(s_i)$

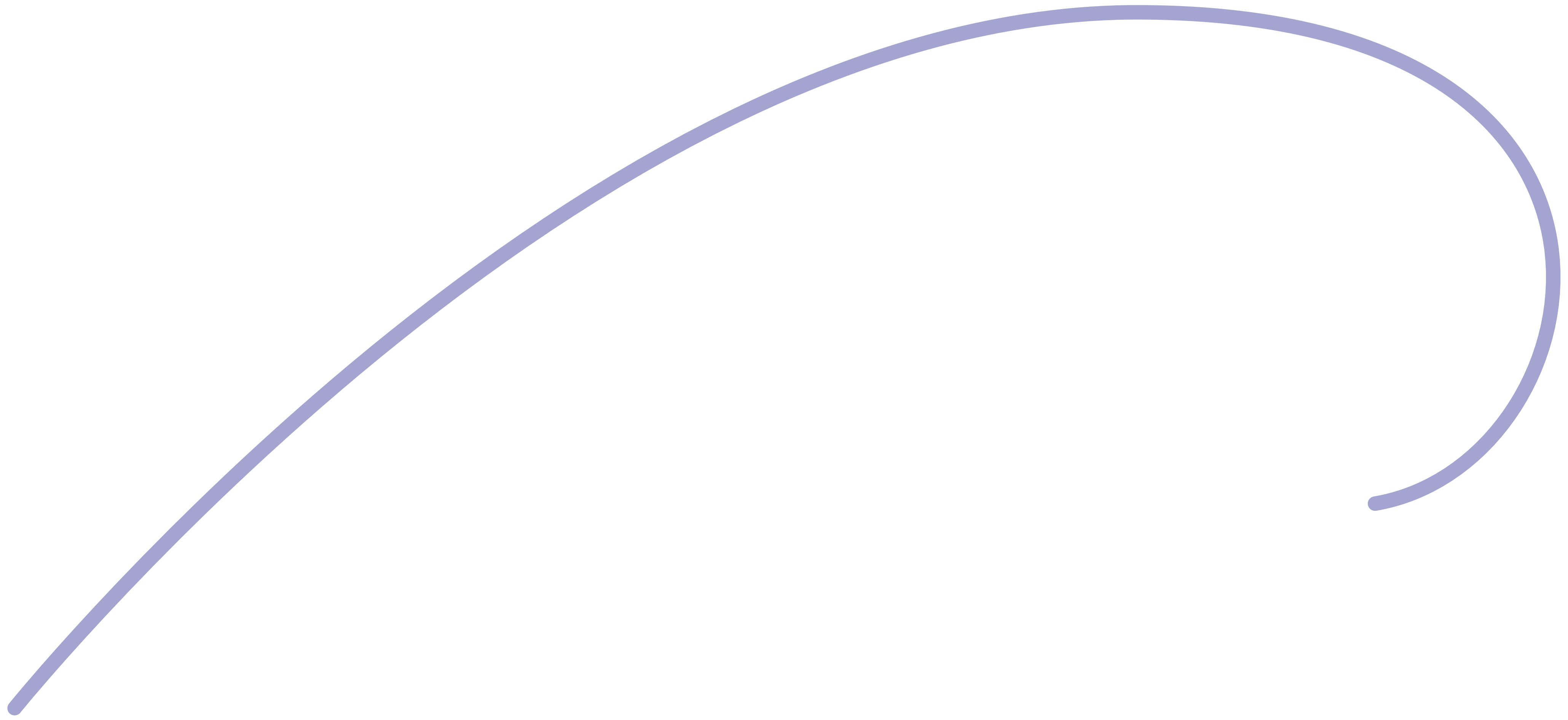
Curvature of a Discrete Curve?

<u>KEY IDEA</u>

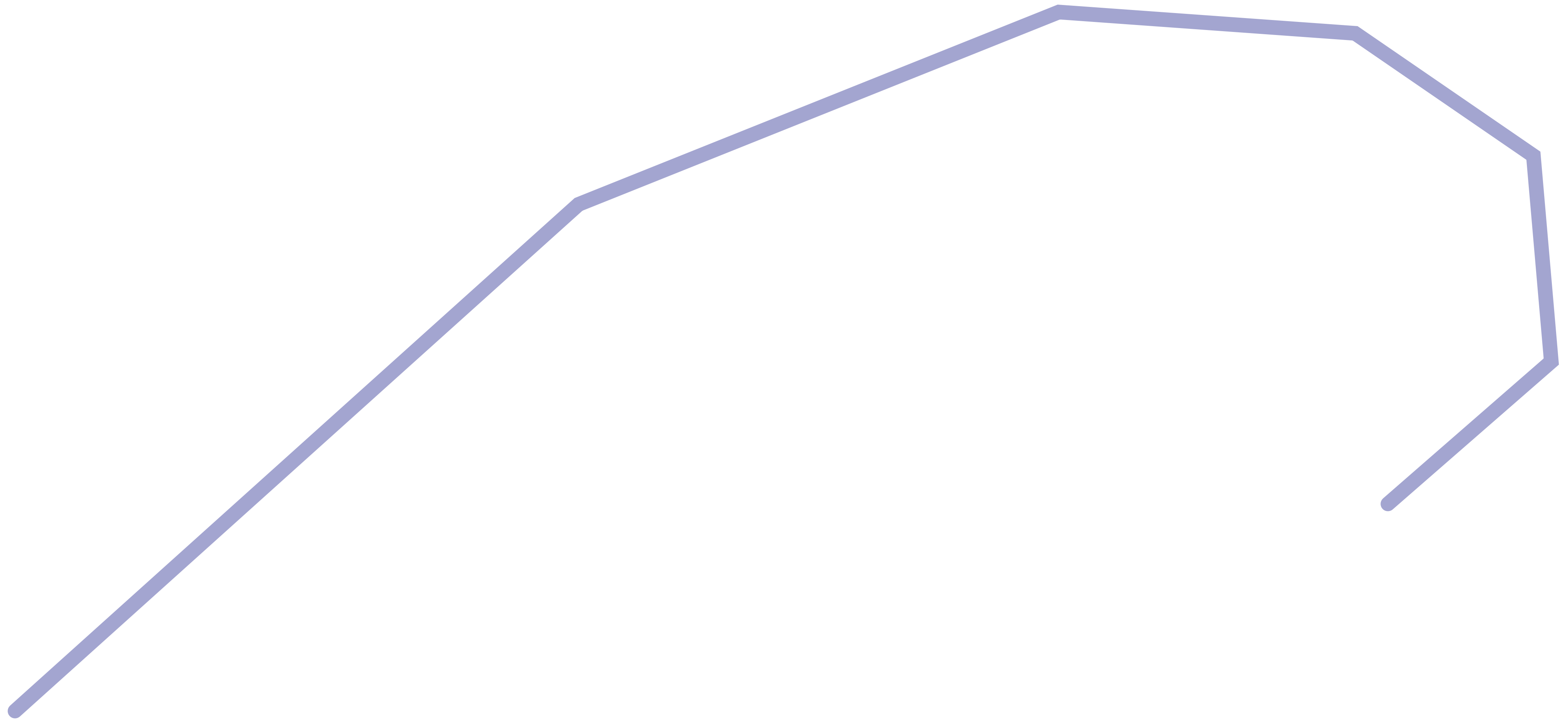
Curvature is a *second derivative*.

Can we directly apply this point of view to a **discrete** curve?

<u>SMOOTH</u>



<u>DISCRETE</u>



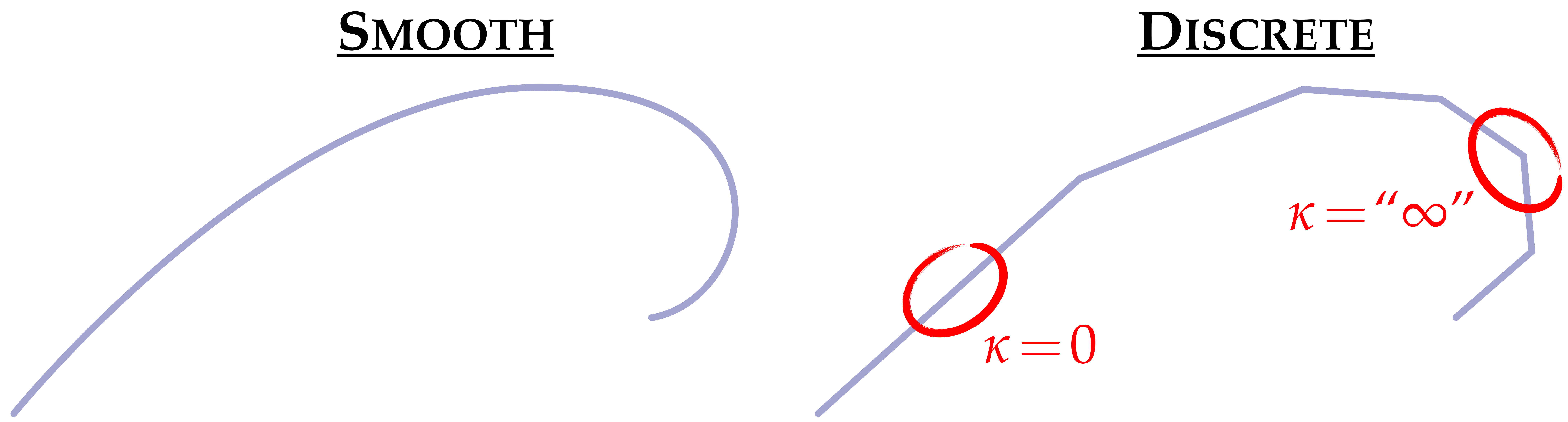
No! Will get either zero or "∞". Need to think about it another way…

What is Discrete Curvature?

KEY IDEA

Curvature is a *second derivative.*

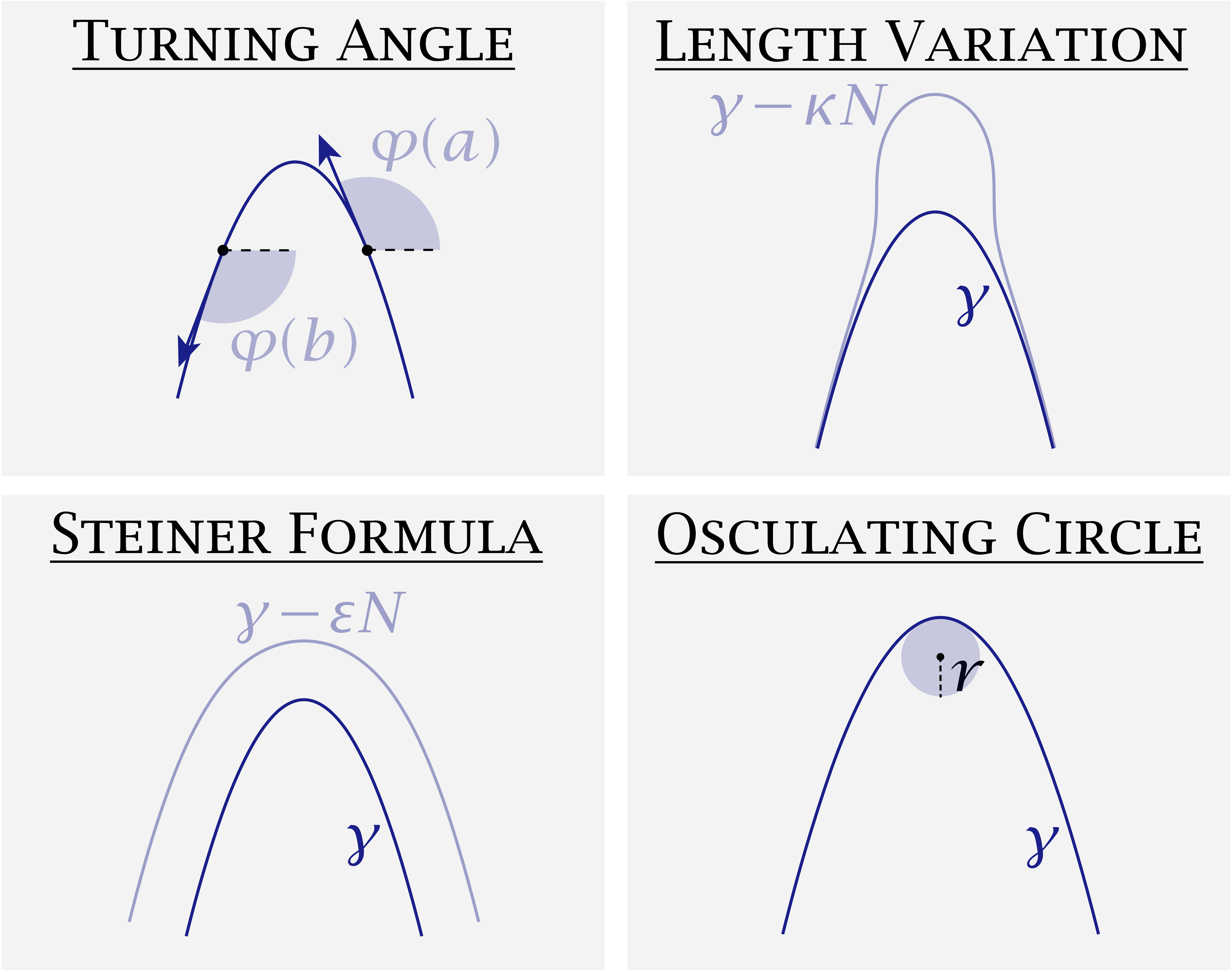
Can we directly apply this point of view to a **discrete** curve?



No! Will get either zero or "∞". Need to think about it another way…

Curvature, Revisited

- In the smooth setting, there are several other **equivalent** definitions of curvature.
- **<u>Idea:</u>** *perhaps some of these definitions can be applied directly to our discrete curve!*
- (Due to time, we will consider just one today; several others are covered in the *Notices* article)

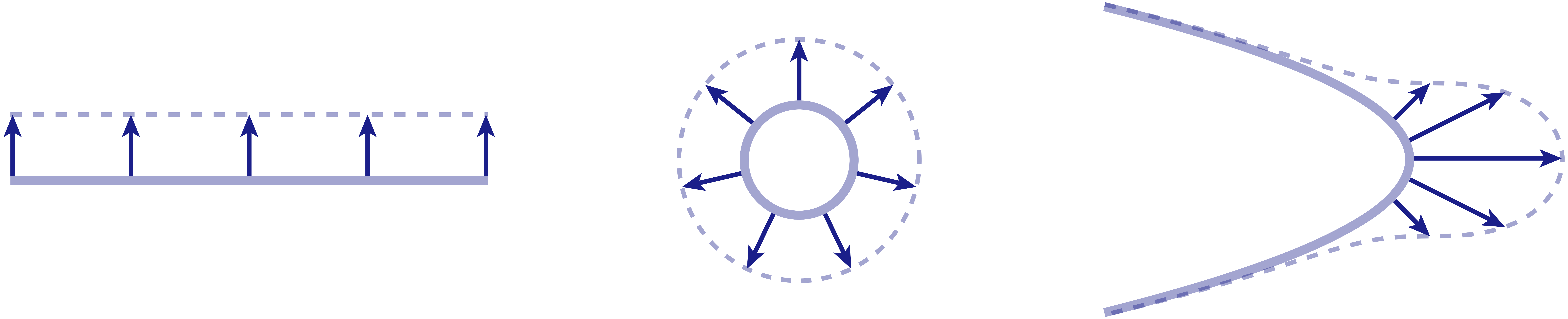


Example: Length Variation

- One way to characterize curvature in smooth setting:

> *The fastest way to increase the length of a curve is to move it in the normal direction, with speed proportional to curvature.*

- **Intuition**: in flat regions, moving the curve doesn't change its length; in curved regions, the change in length (*per unit length*) is large:

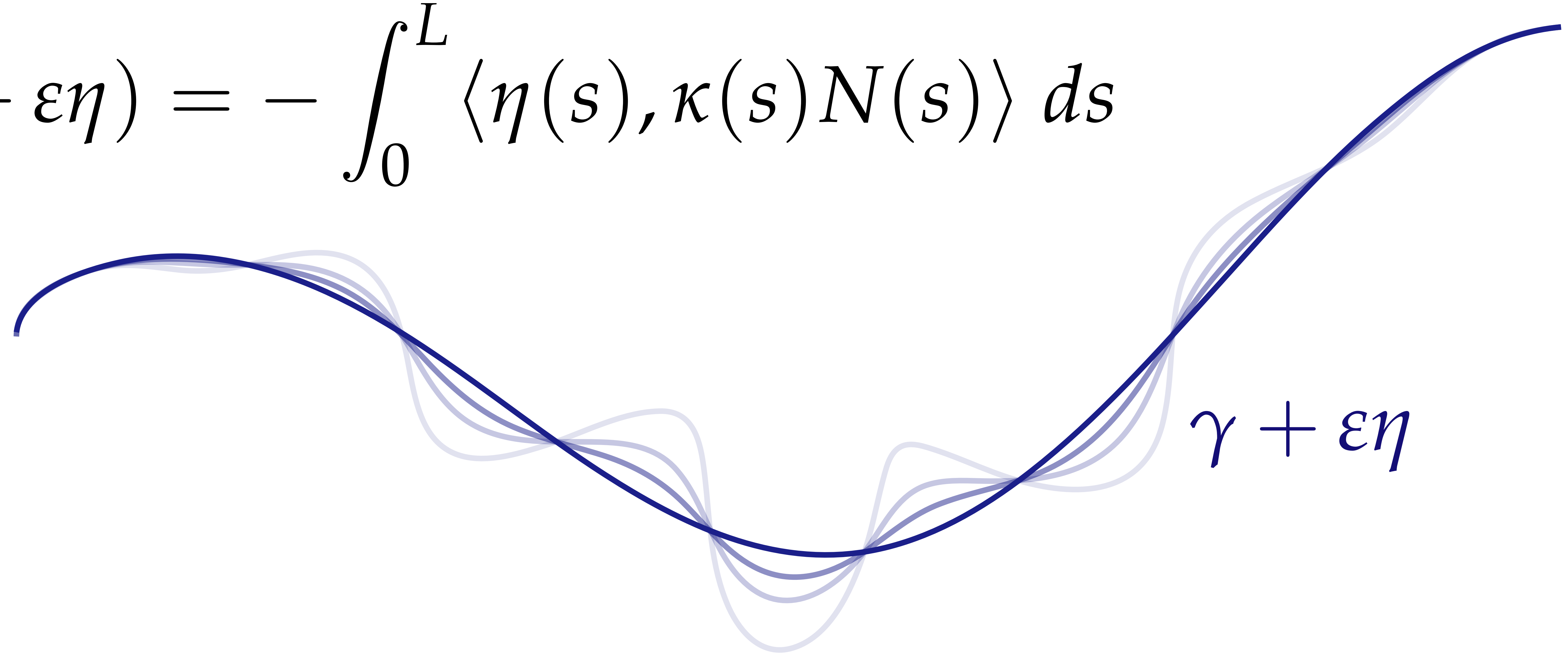


- Discrete curve may not have 2nd derivatives, but certainly has *length!*

Length Variation—Smooth

- More formally, consider an *arbitrary* variation of the curve. *I.e.*, suppose we have another curve* $\eta : [0, L] \to \mathbb{R}^2$. One can show that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \text{length}(\gamma + \varepsilon\eta) = -\int_0^L \langle \eta(s), \kappa(s)N(s) \rangle \, ds$$

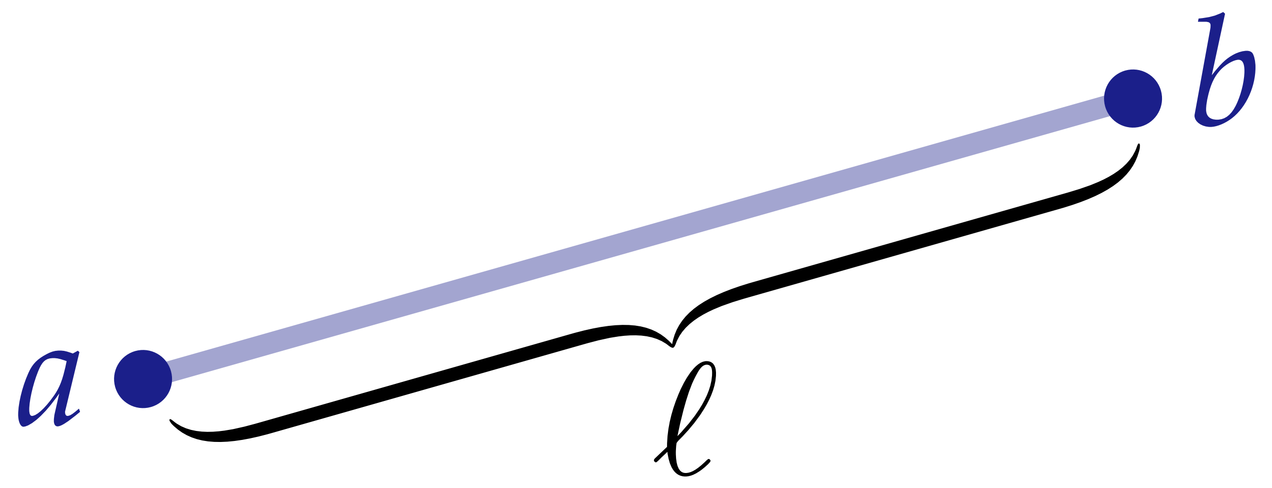


- Therefore, the motion that most quickly decreases length is $\eta = \kappa N$.

*Must go to zero at endpoints (*i.e.*, pass through the origin).

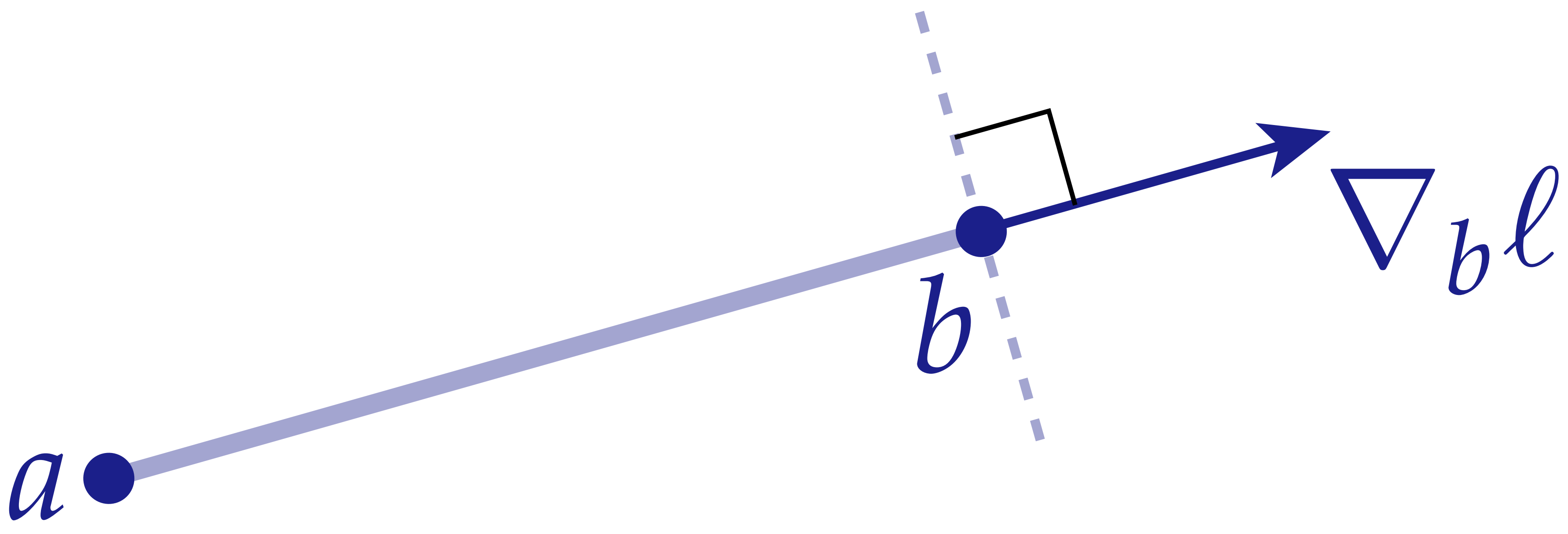
Length Variation — Discrete

- Even simpler in the discrete setting: just take the gradient of length with respect to vertex positions.
- First consider a single line segment:



$$\ell := |b - a|$$

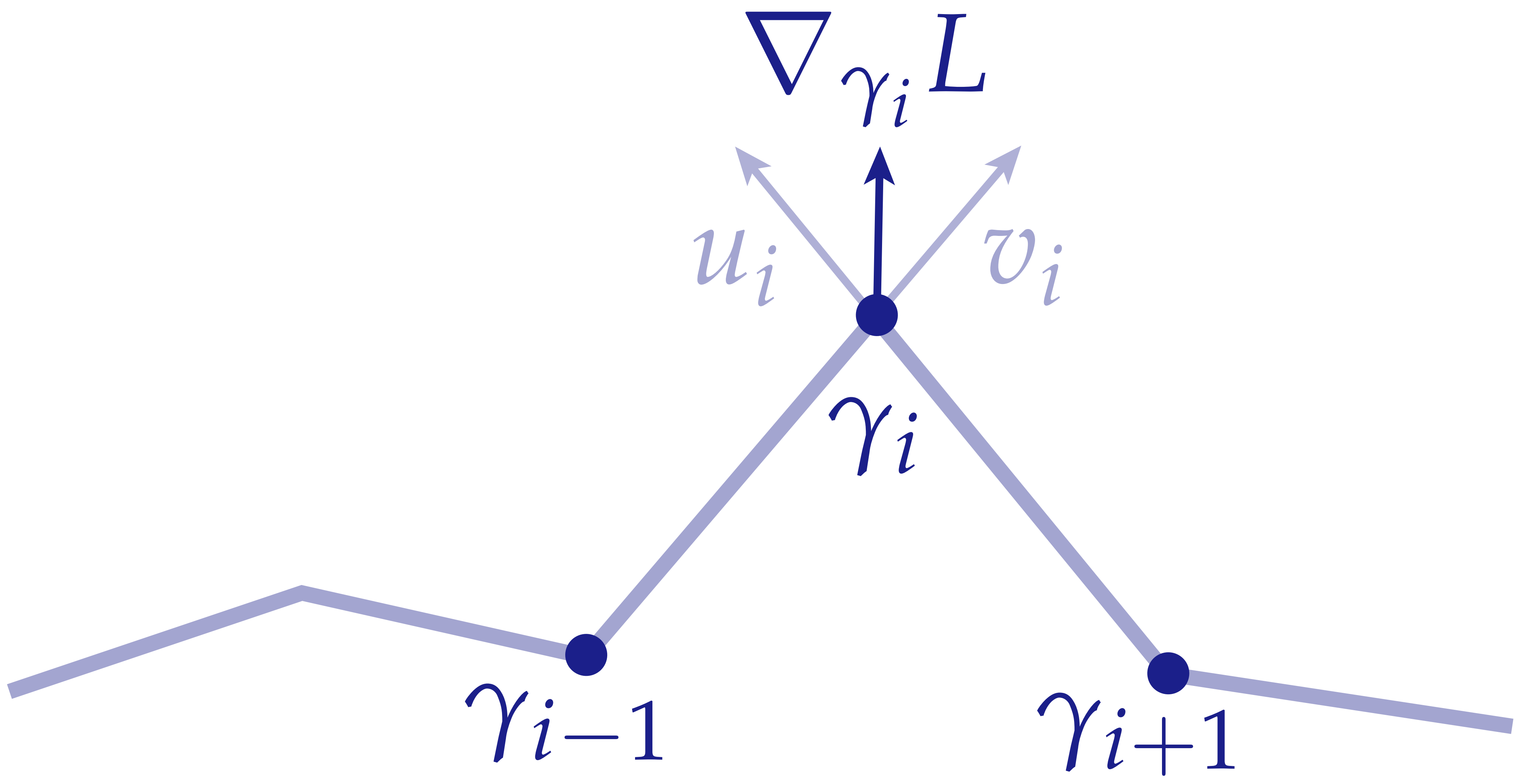
- How can we move the point b to most quickly increase its length?

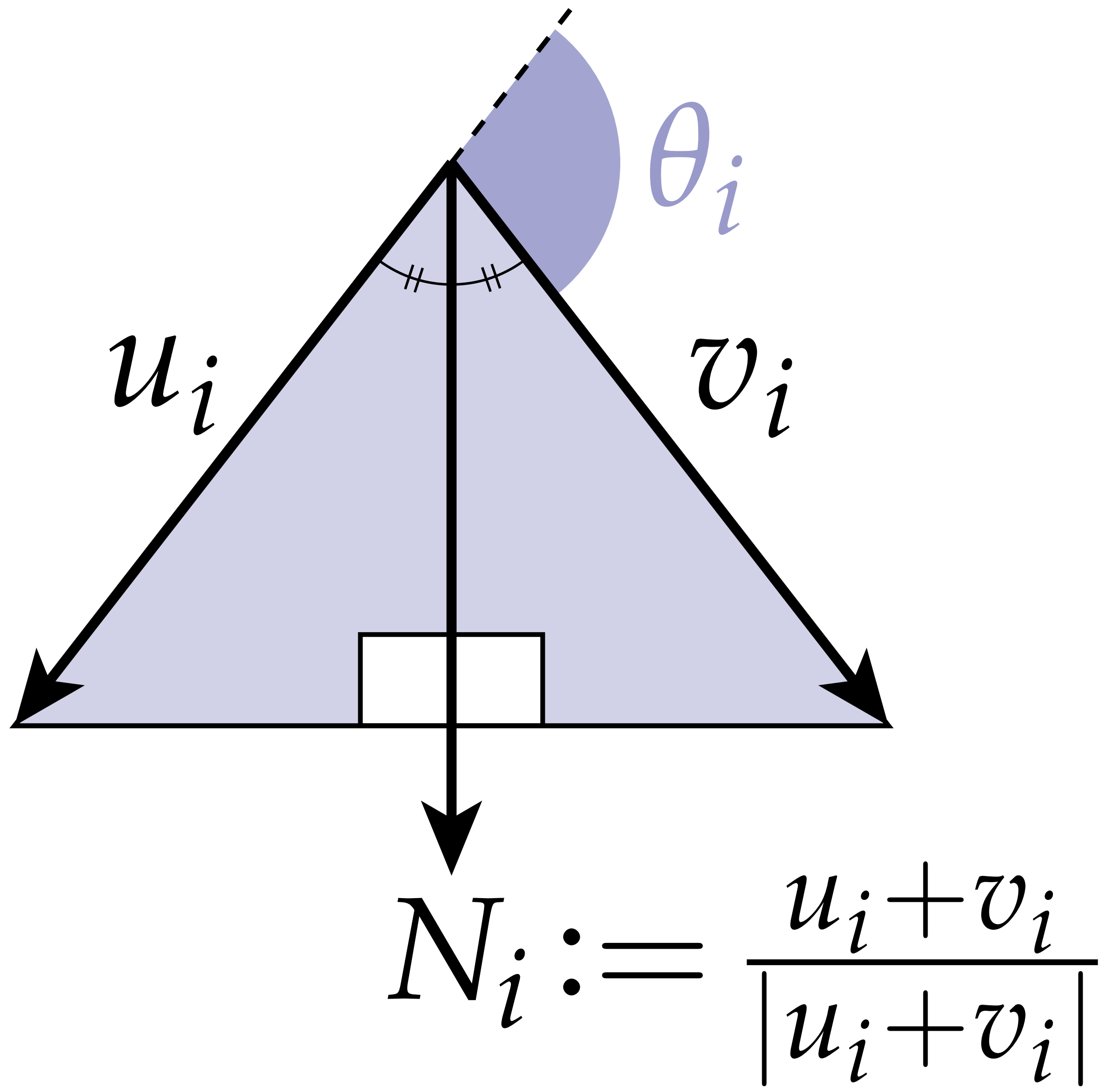


$$\nabla_b \ell = (b - a)/\ell$$

Length Variation—Discrete

- Gradient of total length L with respect to vertex position is just a sum:



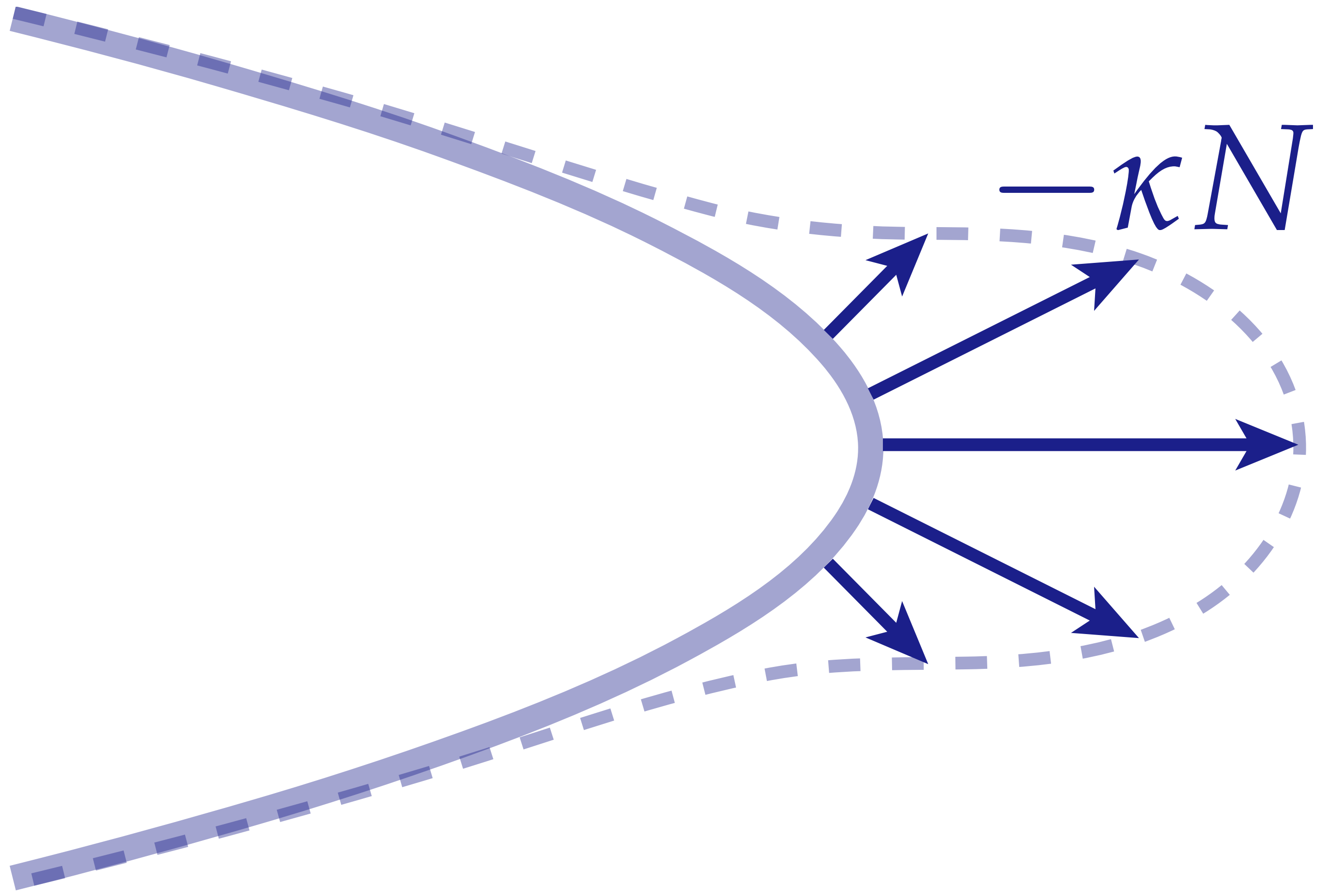


- Can easily re-express in terms of exterior angle θ_i and angle bisector N_i:

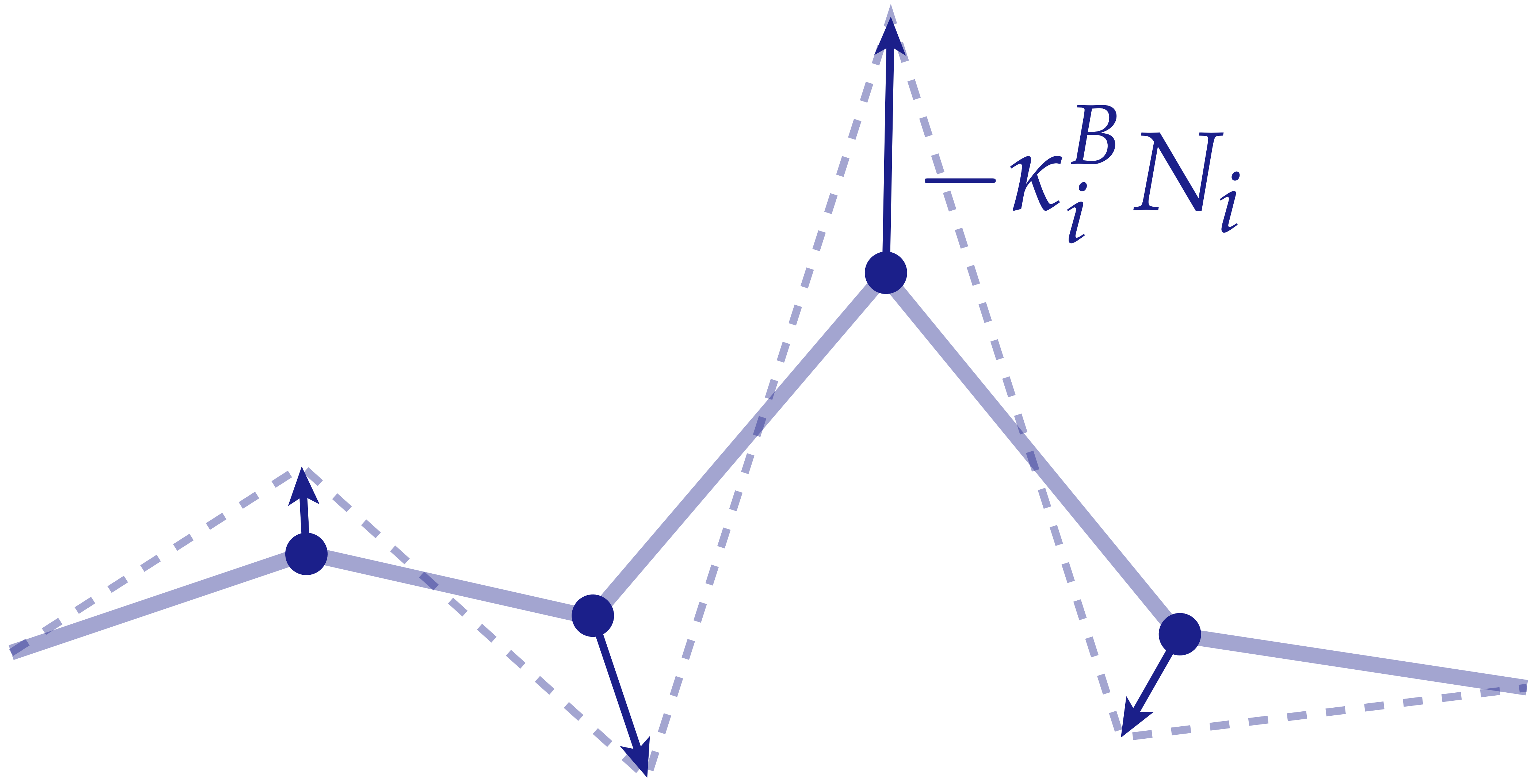
$$\nabla_{\gamma_i} L = 2\sin(\theta_i/2) N_i$$

Discrete Curvature (Length Variation)

- How does this help us define discrete curvature?
- Recall that in the smooth setting, the gradient of length is equal to the curvature times the normal.
- Hence, our expression for the *discrete* length variation provides a definition for the *discrete* curvature times the *discrete* normal.

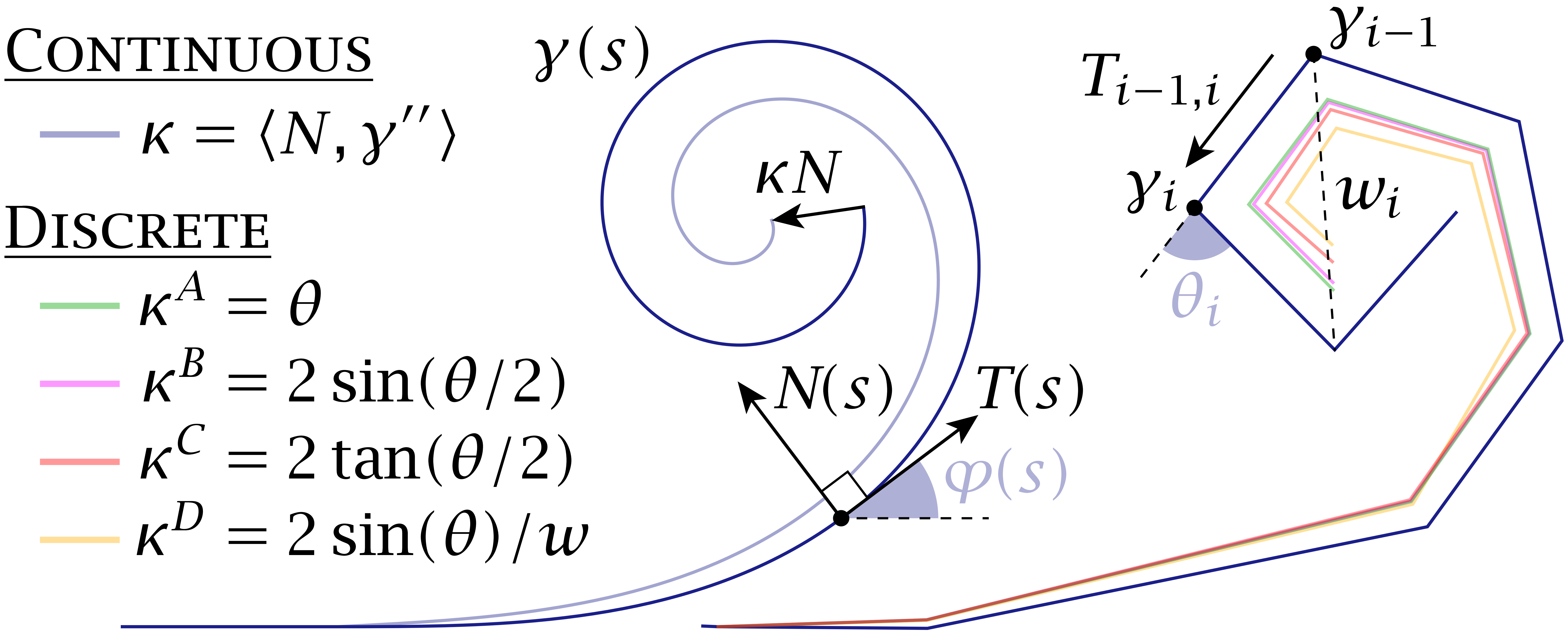


$$\kappa_i^B N_i := 2\sin(\theta_i/2) N_i$$



A Tale of Four Curvatures

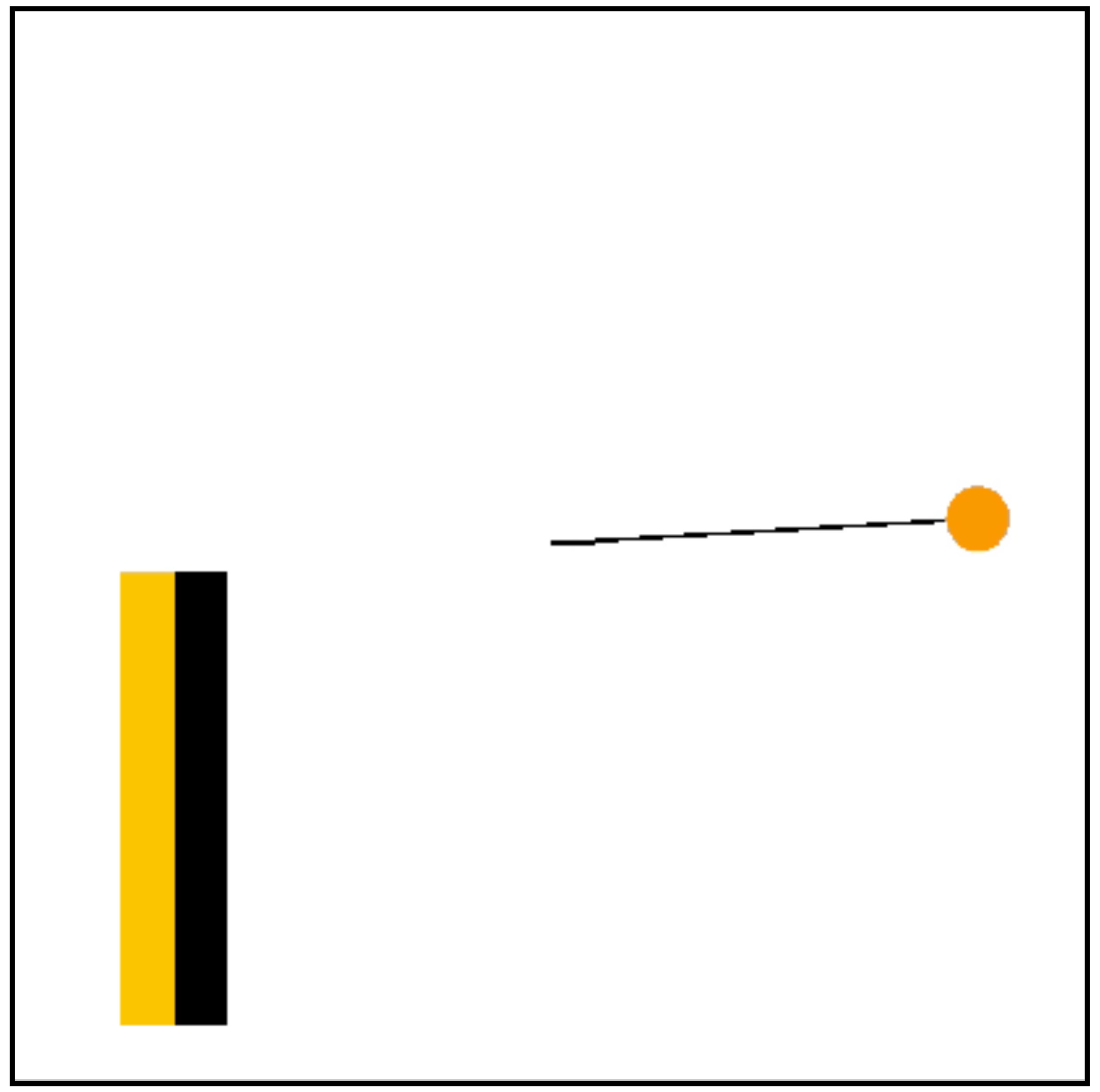
- If we continue this game starting with our four **equivalent** smooth definitions, we will get four **inequivalent** discrete definitions:



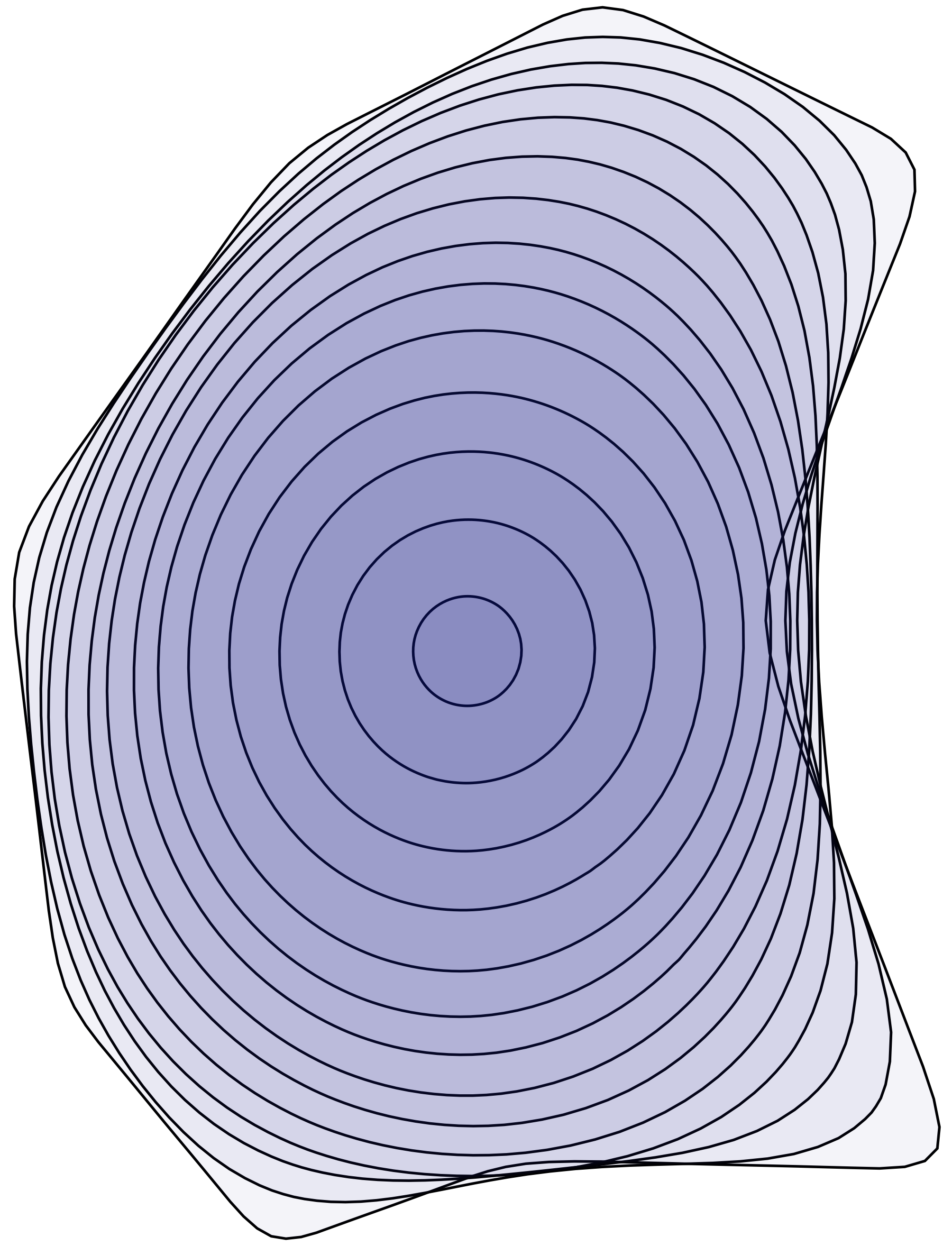
Which one is the "right" definition of discrete curvature?

Pick the Right Tool for the Job

- **Answer:** *pick the right tool for the job!*
- Very rarely one "right" discrete definition
- Each definition plays a role in a different context
- Analogy: in different mechanical problems, we might care about preserving energy but not momentum—or momentum, but not energy.
- Where does this kind of trade-off come up with curvature?



Toy Example: Curvature Flow

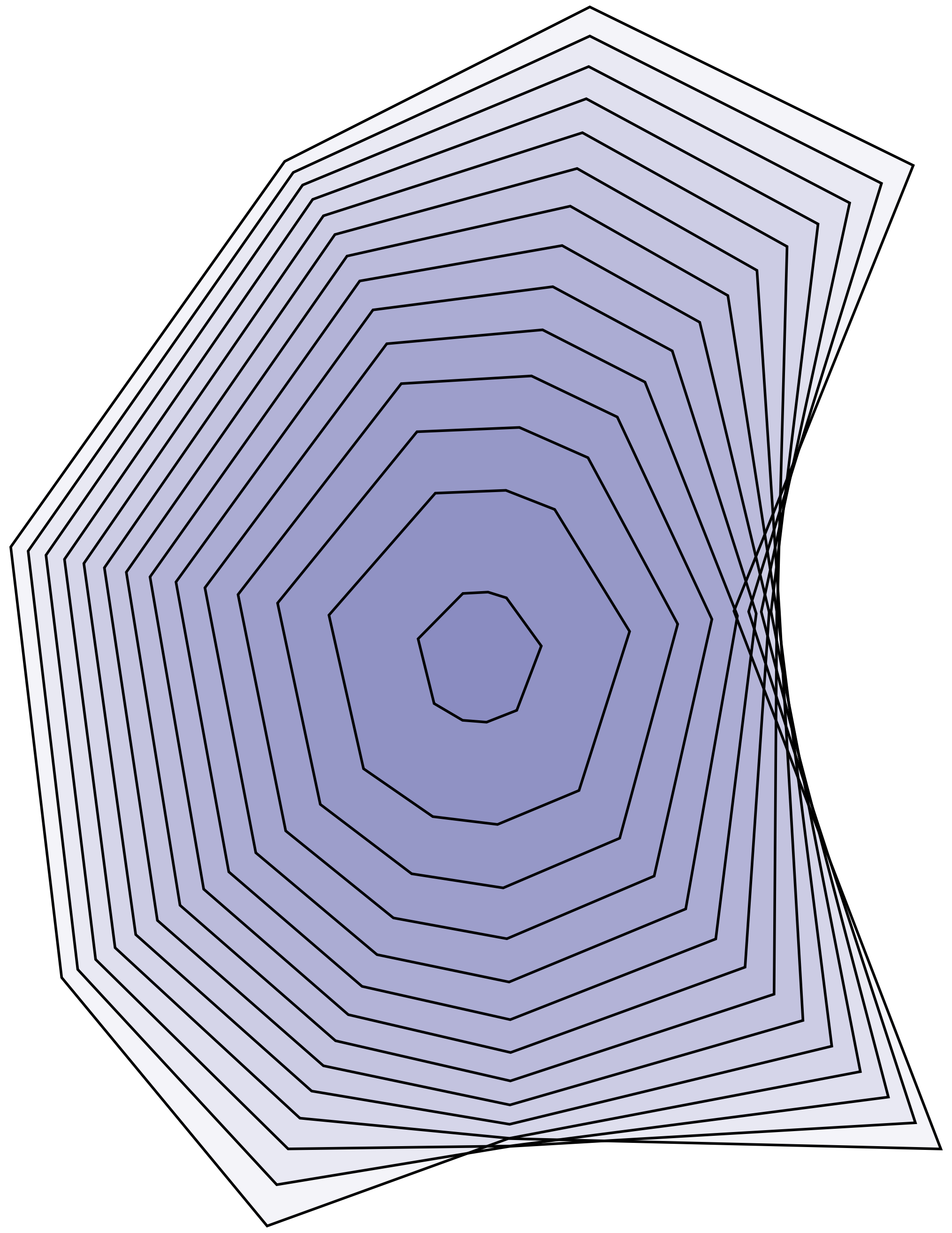


- A classic example is *curvature flow*, where a closed curve moves in the normal direction with speed proportional to curvature:

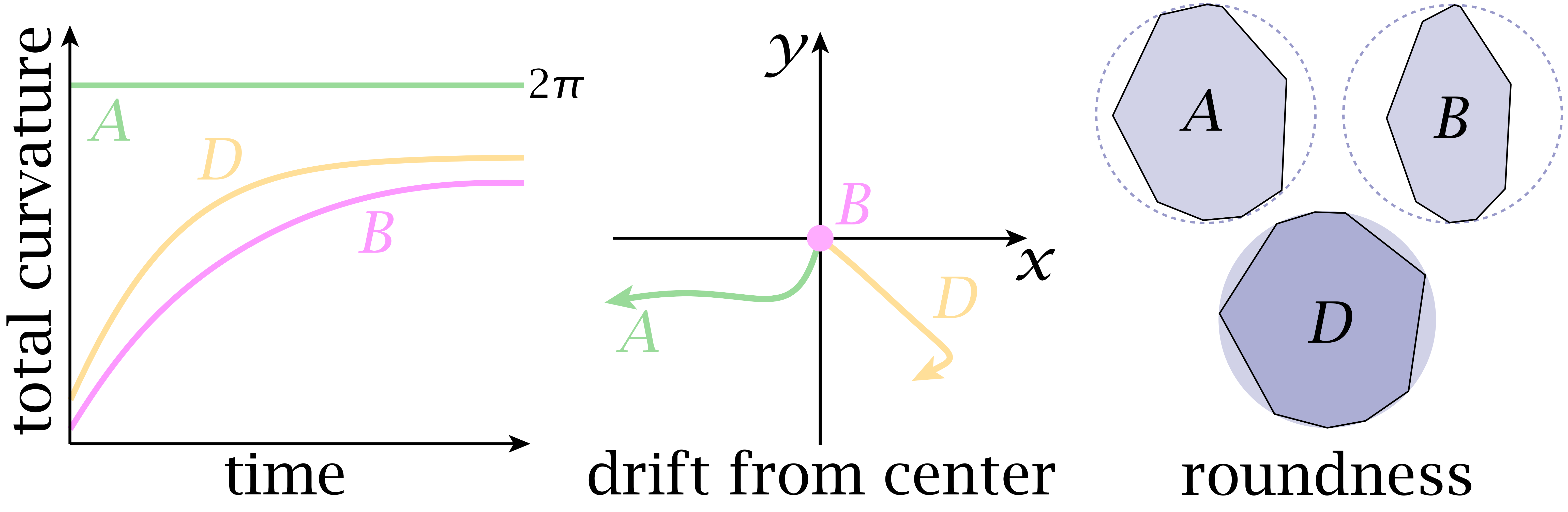
$$\frac{d}{dt}\gamma(s,t) = \kappa(s,t)N(s,t)$$

- Can we construct a discrete curvature flow that faithfully captures the behavior of the smooth flow?
- Some key properties:
  - (**Total**) Total curvature remains constant throughout the flow.
  - (**Drift**) The center of mass does not drift from the origin.
  - (**Round**) Up to rescaling, the flow is stationary for circular curves.

Discrete Curvature Flow—No Free Lunch



- We can approximate curvature flow by repeatedly moving each vertex a little bit in the direction of the discrete curvature normal: $$\gamma_i^{t+1} = \gamma_i^t + \tau \kappa_i N_i$$
- But **no** choice of discrete curvature simultaneously captures all three properties of the smooth flow*:



	Total	Drift	Round
κ^A	✓	×	×
κ^B	×	✓	×
κ^D	×	×	✓

*In fact, it's impossible!

Properties of Smooth Curvature

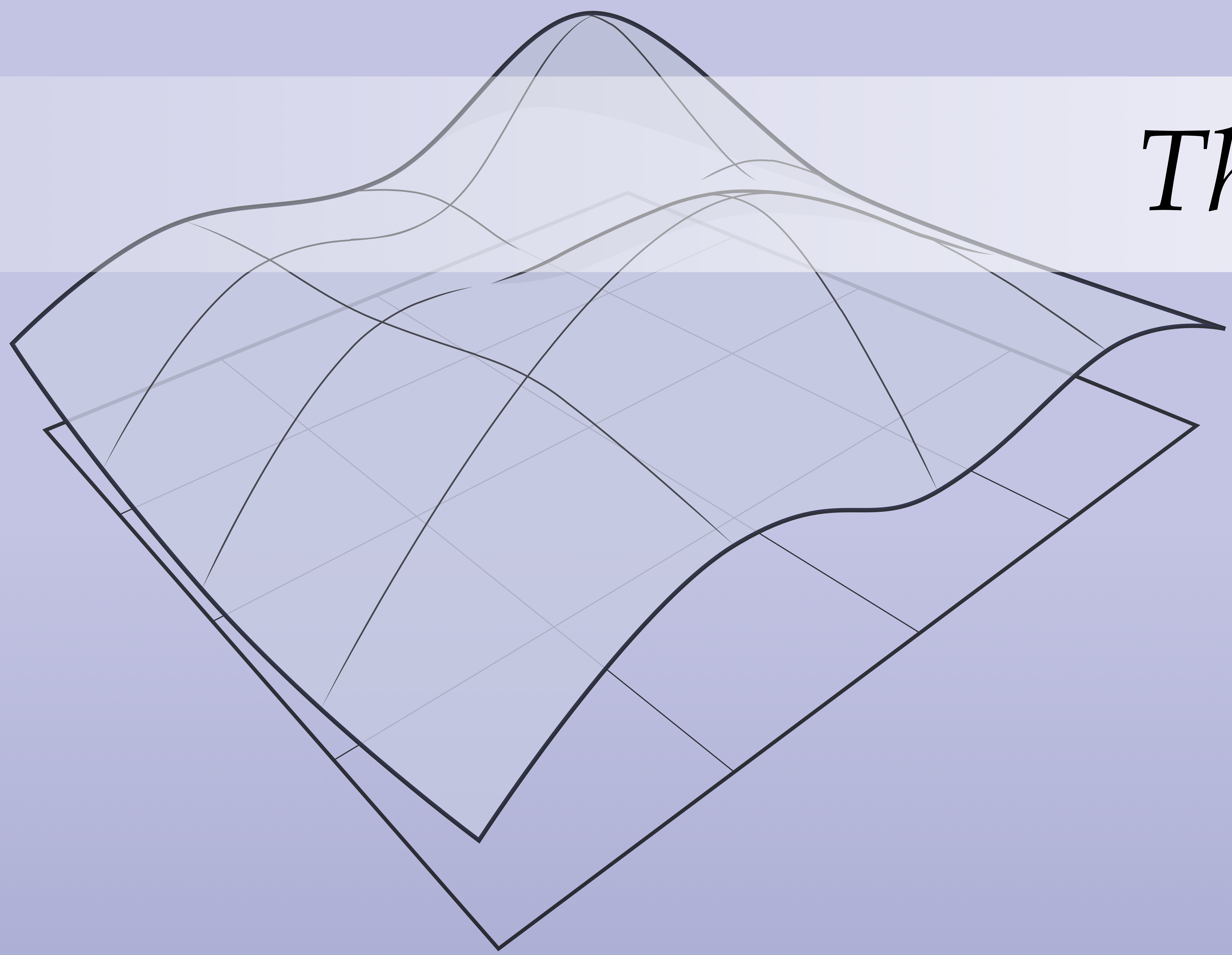
What are other properties of curvature we might like to preserve?

- **locality** (depends only on a small piece of the curve)
- **coordinate invariance** (unchanged by rigid rotations & translations)
- **topological invariance** (for closed loop, integrates to multiple of 2π)
- **changes sign under reflections** (hence zero for straight lines)
- **scaling inverse to scaling of curve** (hence goes to ∞ for small circles)
- …

Food for thought: to what degree do such axioms determine a definition?

Coming up next…

- Beyond this "toy" problem, the *no free lunch* scenario is quite common in discrete differential geometry.
  - *E.g.*, Whitney-Graustein / Kirchoff analogy for curves; conservation of energy, momentum, and symplectic form for conservative time integrators; discrete Laplace operators…
- More generally, **The Game** played in DDG often leads to new & unexpected ways of thinking about geometry, and formulating geometric algorithms. (E.g., faster, simpler, clearer guarantees, …)
- Will see *much* more of this as the course continues!



Thanks!

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DISCRETE DIFFERENTIAL GEOMETRY

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