Discrete Parametric Surfaces

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Short Course on Discrete Differential Geometry

San Diego, Joint Mathematical Meetings 2018

Overview

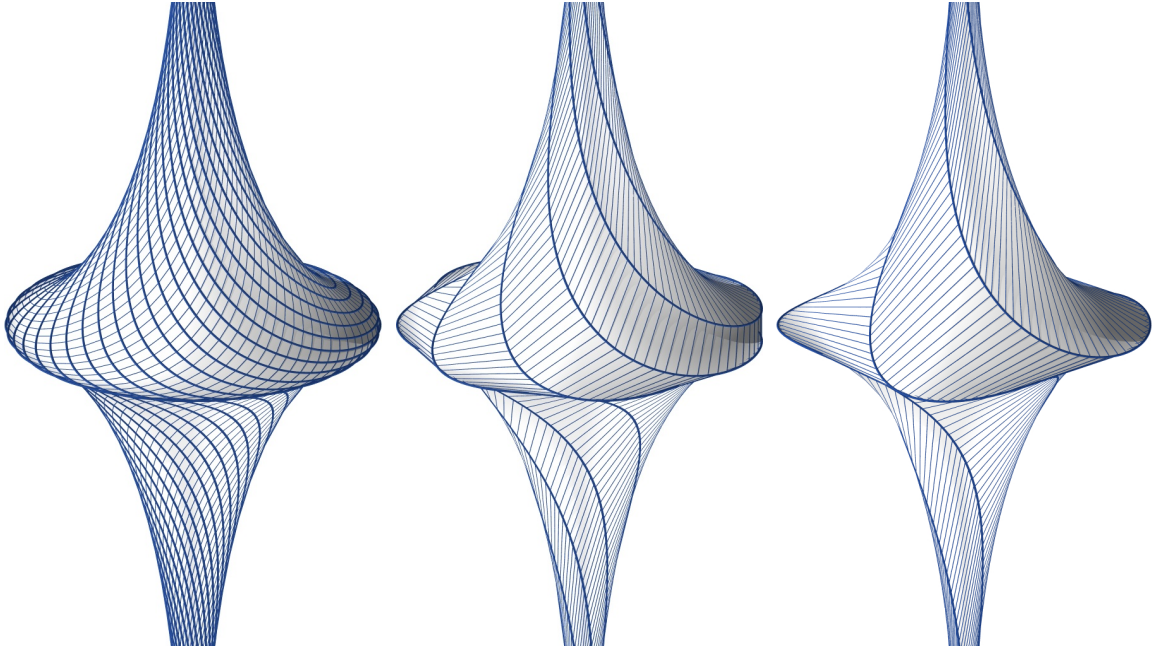
- Introduction and Notation
- An integrable 3-system: circular surfaces
- An integrable 2-system: K-surfaces
- Computing minimal surfaces
- Freeform architecture

Part I

Introduction and Notation

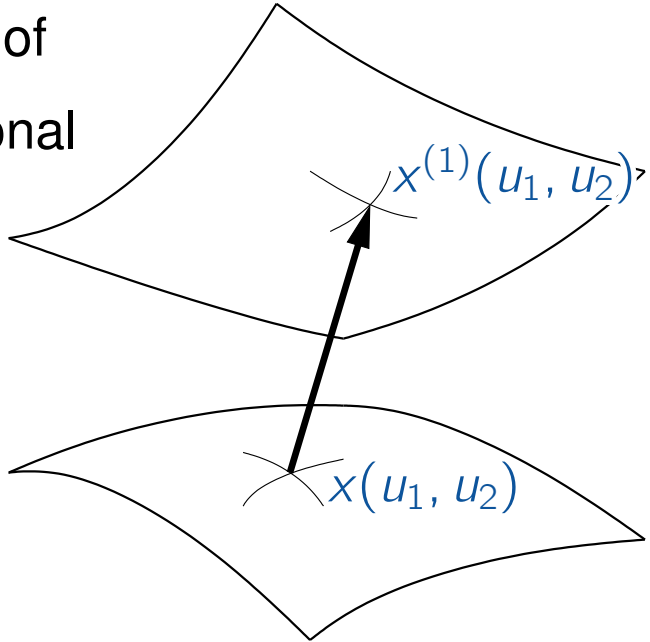
Parametric Surfaces

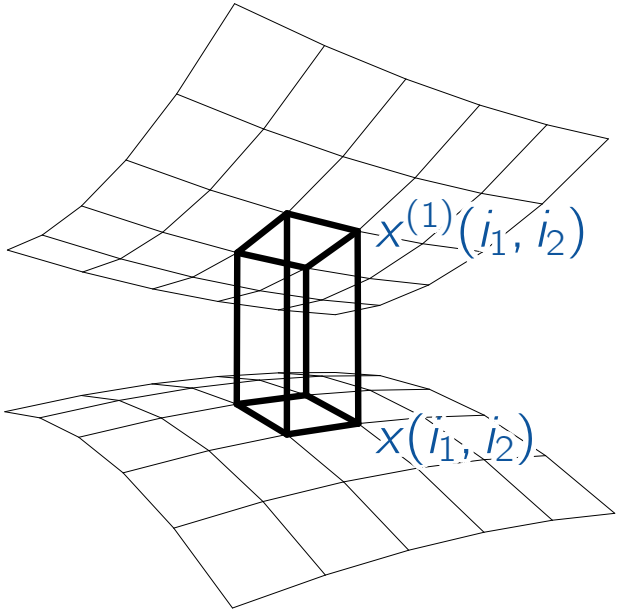
- Surfaces $x(u)$ where $u \in \mathbb{Z}^k$ or $u \in \mathbb{R}^k$ or $u \in \mathbb{Z}^k \times \mathbb{R}^l$
- continuous case, discrete case, mixed case



Parametric Surfaces

- Surfaces $x(u)$ where $u \in \mathbb{Z}^k$ or $u \in \mathbb{R}^k$ or $u \in \mathbb{Z}^k \times \mathbb{R}^l$
- continuous case, discrete case, mixed case
- Transformations of $(k-1)$-dimensional surfaces yield k-dim. surfaces





Examples of discrete surfaces

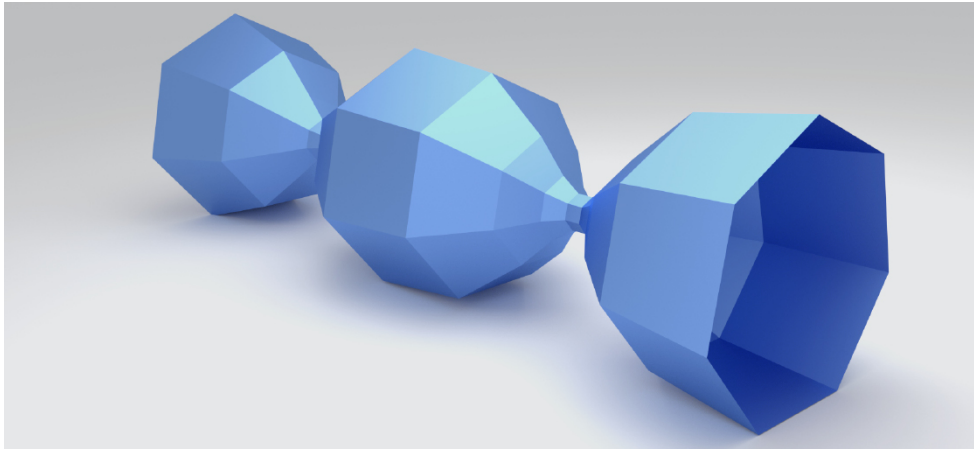
- quad meshes in freeform architecture

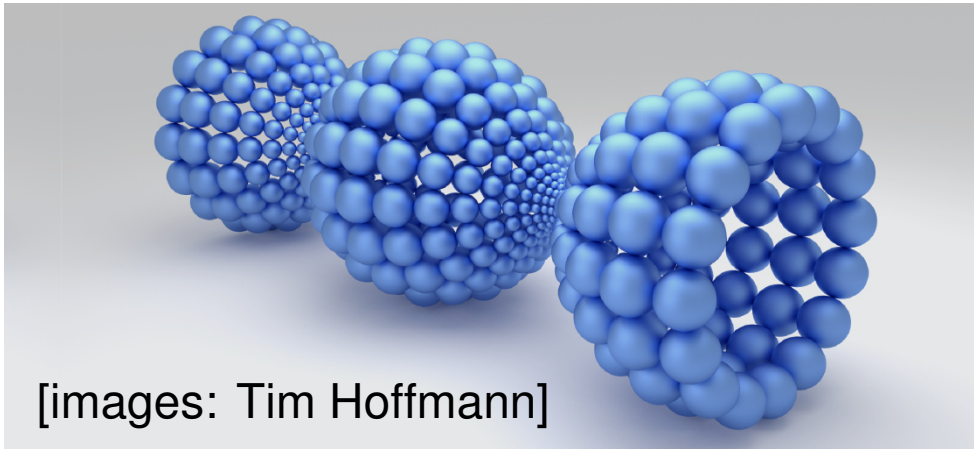

[Yas Marina Hotel, Abu Dhabi]


[Hippo house, Berlin zoo]

Examples of discrete surfaces

- various mappings
  $\mathbb{Z}^k \to \{\text{points}\}$ or
  $\mathbb{Z}^k \to \{\text{spheres}\}$ or ...
- Discrete minimal surfaces,
  discrete cmc surfaces,
  discrete K-surfaces,
  etc.





[images: Tim Hoffmann]

Notation

- discrete parametric surface $\cong$ net
- Right shift of a surface x in the k-th direction is denoted by x_k

  Left shift of a surface x in the k-th direction is denoted $x_{\bar{k}}$

$$\begin{cases} x_1(k,l) = x(k+1,l) \\ x_2(k,l) = x(k,l+1) \end{cases}$$

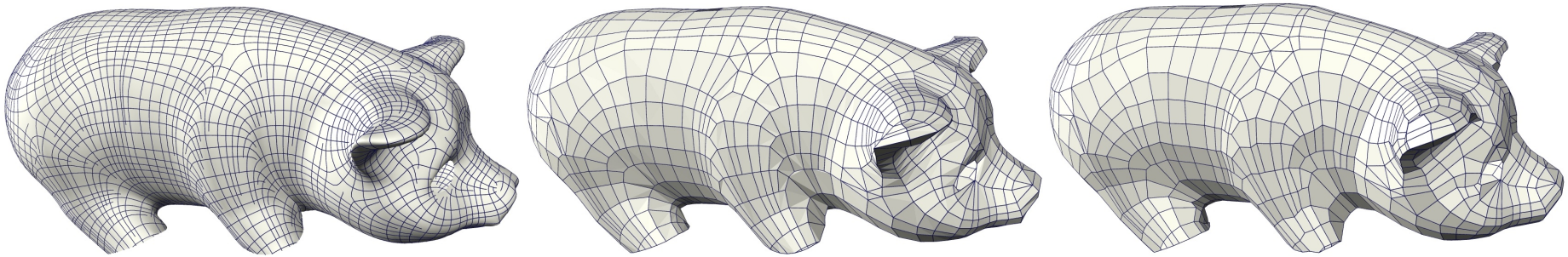
$$\begin{tikzcd}
x_{\bar{1}2} \arrow[r, dash] \arrow[d, dash] & x_2 \arrow[r, dash] \arrow[d, dash] & x_{12} \arrow[d, dash] \\
x_{\bar{1}} \arrow[r, dash] \arrow[d, dash] & x \arrow[r, dash] \arrow[d, dash] & x_1 \arrow[d, dash] \\
x_{\bar{1}\bar{2}} \arrow[r, dash] & x_{\bar{2}} \arrow[r, dash] & x_{1\bar{2}}
\end{tikzcd}$$

- Differences are used instead of derivatives $\Delta_i x = x_i - x$

Surface classes: conjugate surfaces

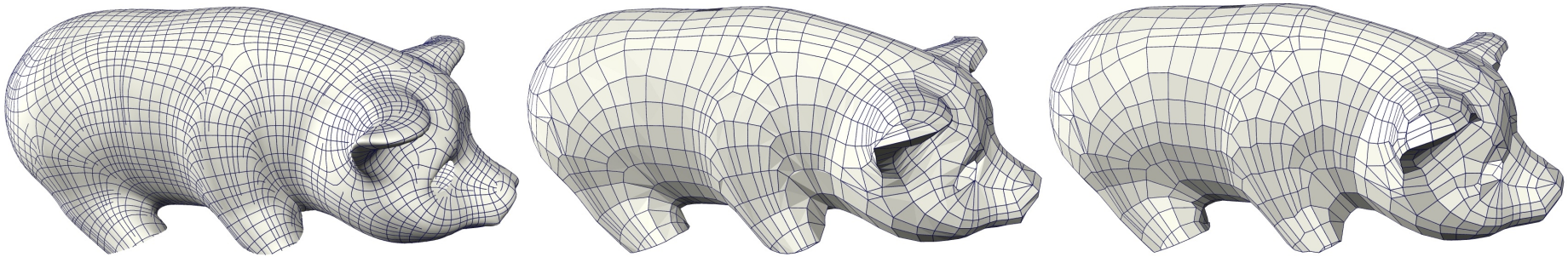
- a smooth conjugate surface $x(u)$ has "planar" infinitesimal quads

$$
\begin{aligned}
&3\,\mathrm{vol}\big(\,\textbf{c.h.}\,\big(x(u),\ x(u_1{+}\varepsilon,u_2),\ x(u_1,u_2{+}\varepsilon),\ x(u_1{+}\varepsilon,u_2{+}\varepsilon)\big)\big)/\varepsilon^4\\
&= \det(x(u_1{+}\varepsilon,u_2)-x(u),\ x(u_1,u_2{+}\varepsilon)-x(u),\ x(u_1{+}\varepsilon,u_2{+}\varepsilon)-x(u))\frac{1}{2\varepsilon^4}\\
&\approx \det(\varepsilon\partial_1 x,\ \varepsilon\partial_2 x,\ \varepsilon\partial_1 x+\varepsilon\partial_2 x+2\varepsilon^2\partial_{12}x)/2\varepsilon^4\\
&= \det(\partial_1 x,\partial_2,\partial_{12}x)=0
\end{aligned}
$$



Surface classes: conjugate surfaces

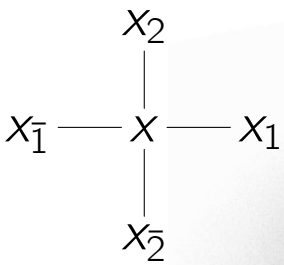
- a discrete conjugate surface $x(u)$ has planar elementary quads
- $\det(\Delta_k x, \Delta_l x, \Delta_{kl} x) = 0 \iff$
- $\begin{array}{ccc} x_l & — & x_{kl} \\ | & & | \\ x & — & x_k \end{array}$ is co-planar for all $l, k \iff$ (generically)
- there are coefficient functions c^{lk}, c^{kl} s.t. $\Delta_k \Delta_l x = c^{lk} \Delta_k x + c^{kl} \Delta_l x$.

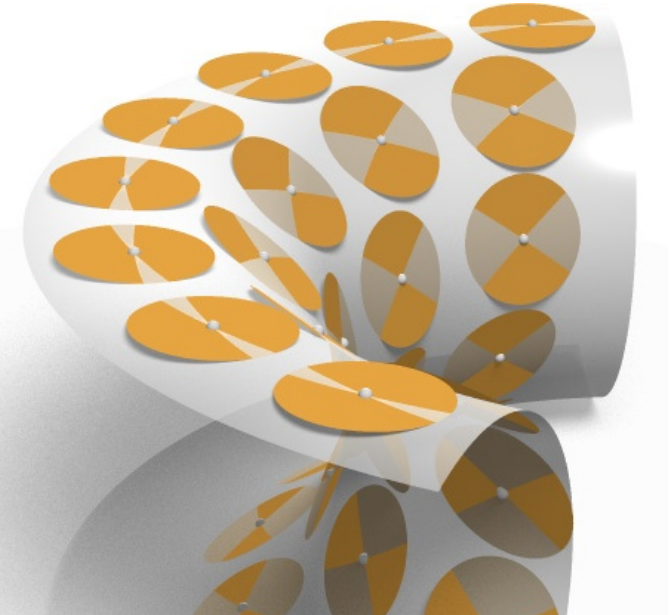


Surface classes: asymptotic surfaces

- Parameter lines are asymptotic, i.e., in every point they indicate the intersection of the saddle-shaped surface with its own tangent plane
- smooth case: $\partial_{11}x, \partial_{22}x \in \mathrm{span}\{\partial_1 x, \partial_2\}$.
- discrete case:

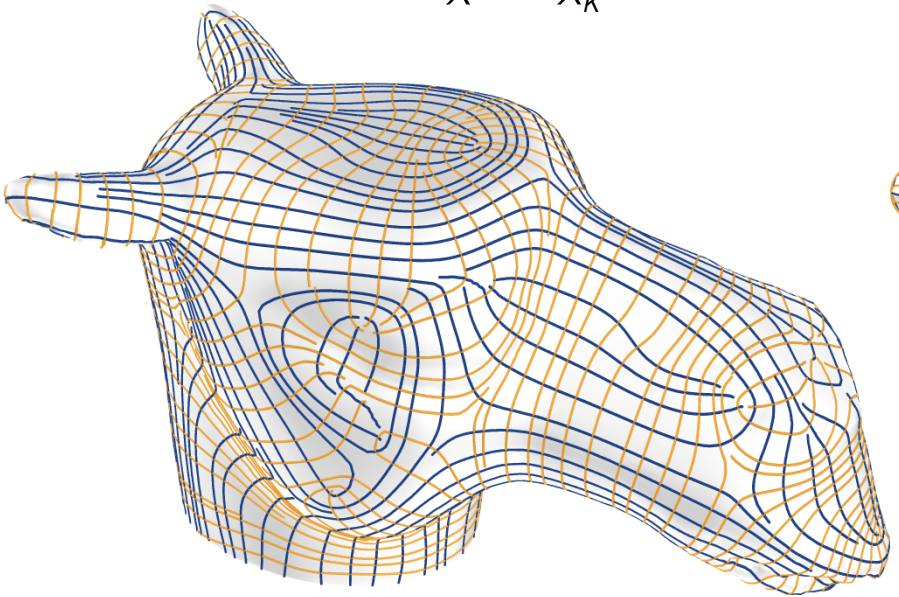
$x, x_1, x_{\bar{1}}, x_2, x_{\bar{2}} \in$ plane $P(u)$

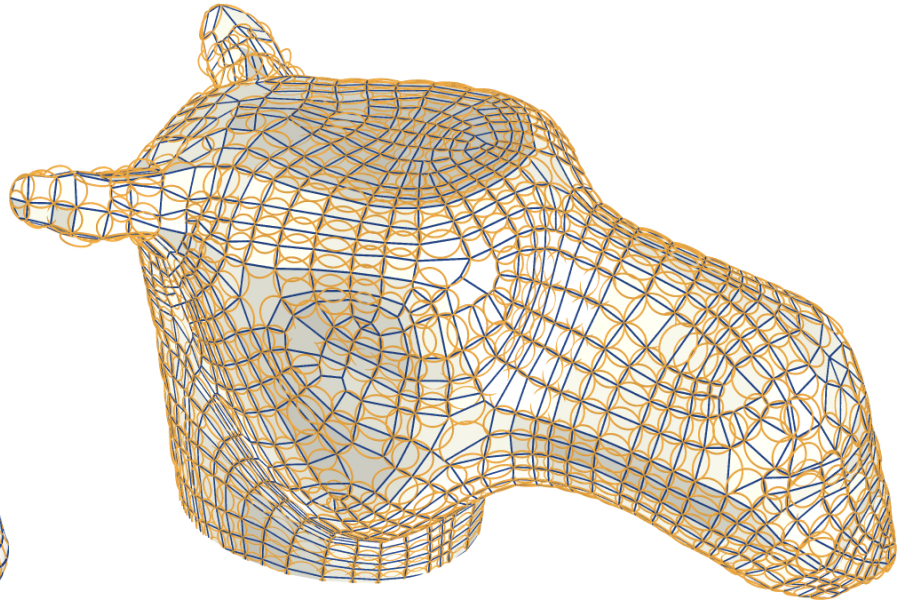




Surface classes: principal surfaces

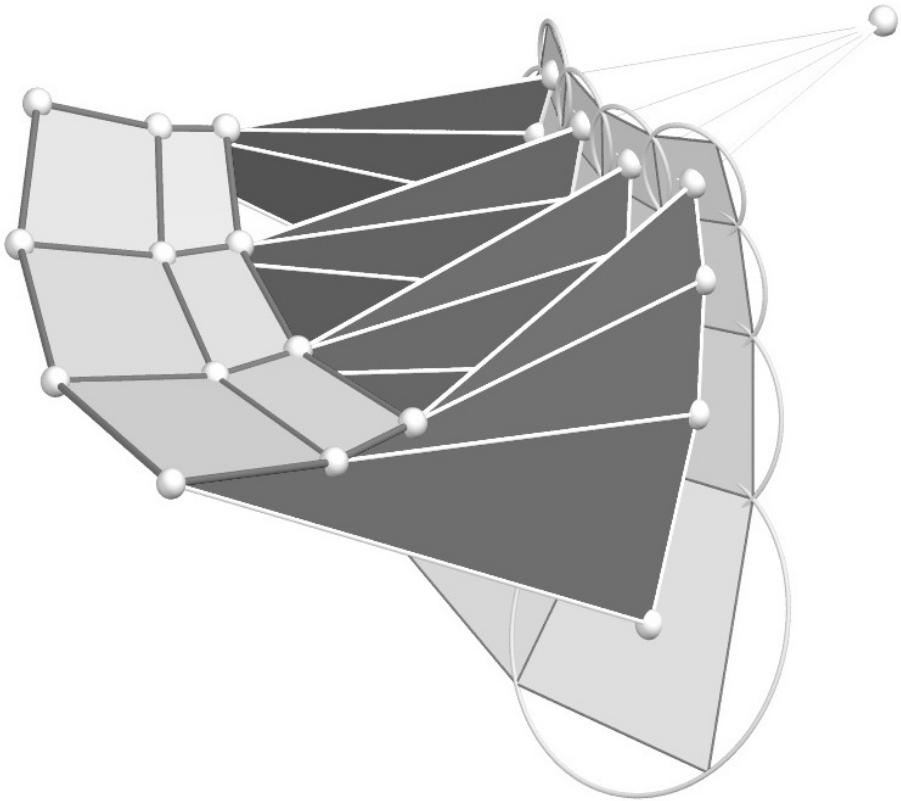
- smooth case: parameter lines are conjugate + orthogonal
- discrete case: $\begin{array}{ccc} x_l & — & x_{kl} \\ | & & | \\ x & — & x_k \end{array}$ is circular for all l, k





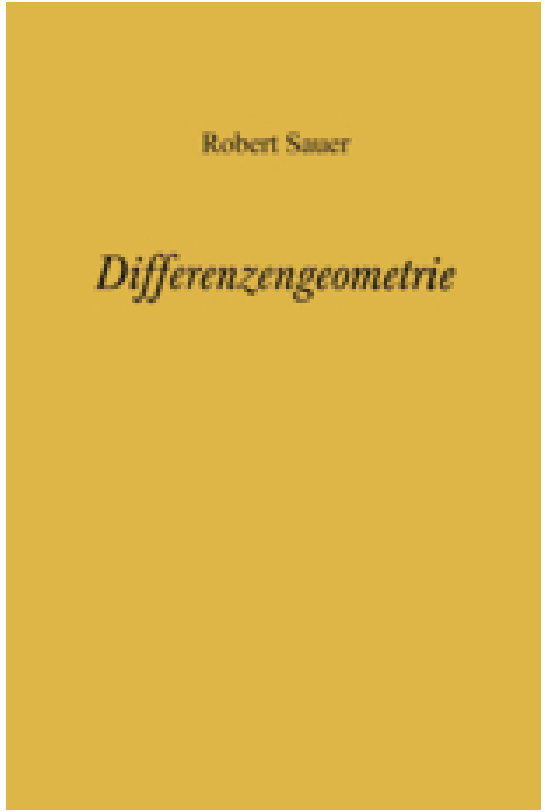
Why is circularity “principal”?

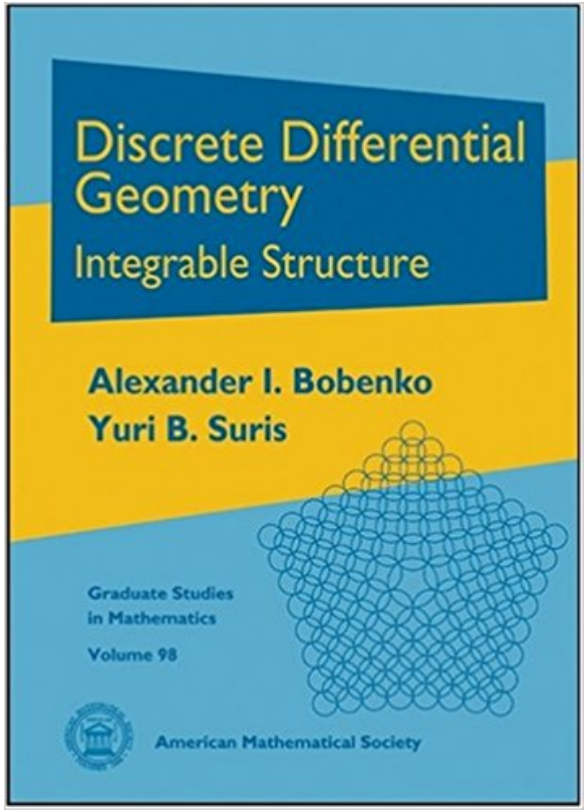
- smooth case: principal curves are characterized by developabililty of the surface formed by normals
- discrete case: normals are circles’ axes. Observe developability
- (convergence can be proved)



Monographs

- [R. Sauer: Differenzengeometrie, Springer 1970]
- [A. Bobenko, Yu. Suris: Discrete Differential Geometry, AMS 2009]



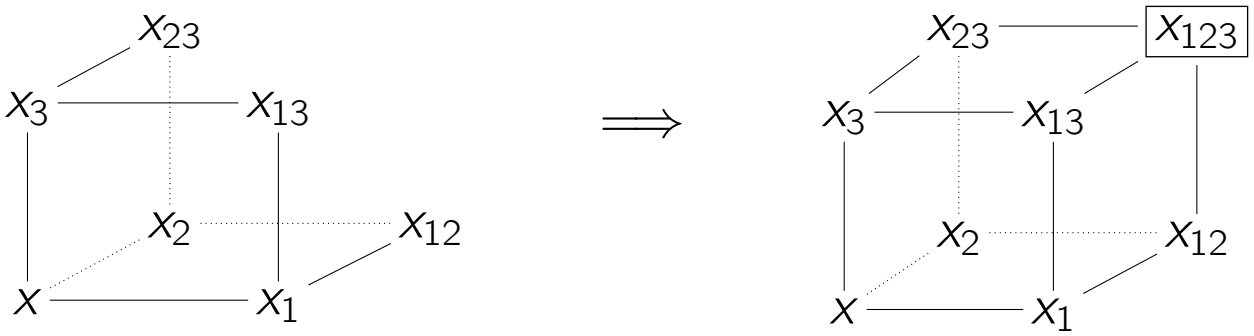


Part II

3-systems

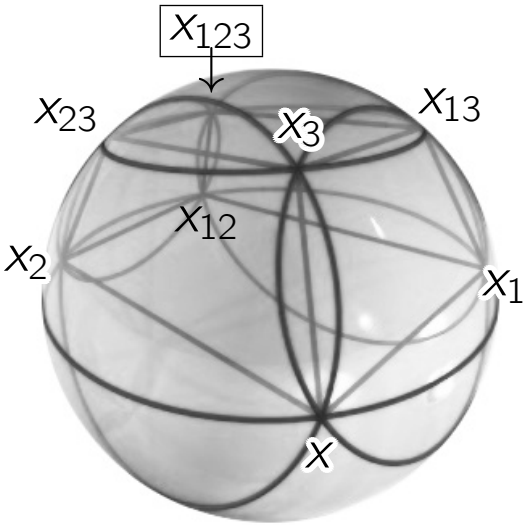
Conjugacy as a 3-system

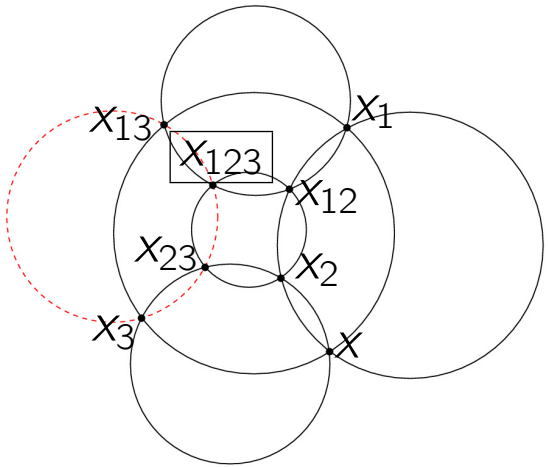
- Generically, a conjugate net $x(u_1, u_2, u_3)$ is uniquely determined by arbitrary initial values $x(0, u_2, u_3)$ and $x(u_1, 0, u_3)$ and $x(u_1, u_2, 0)$.
- Proof is trivial (intersect planes)

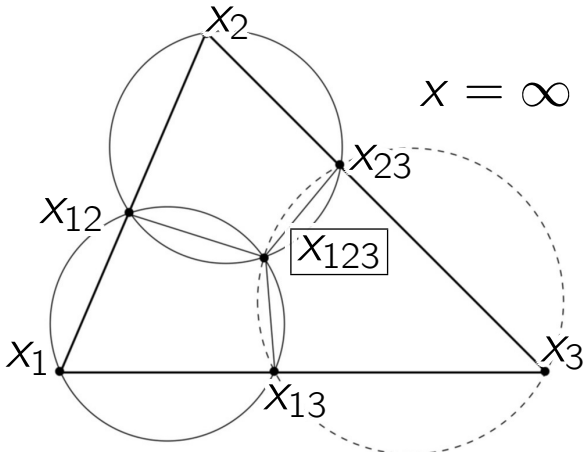


Circularity as a 3-system

- Generically, a circular net $x(u_1, u_2, u_3)$ is uniquely determined by arbitrary initial values $x(0, u_2, u_3)$ and $x(u_1, 0, u_3)$ and $x(u_1, u_2, 0)$.
- Proof is not trivial (Miquel's theorem)





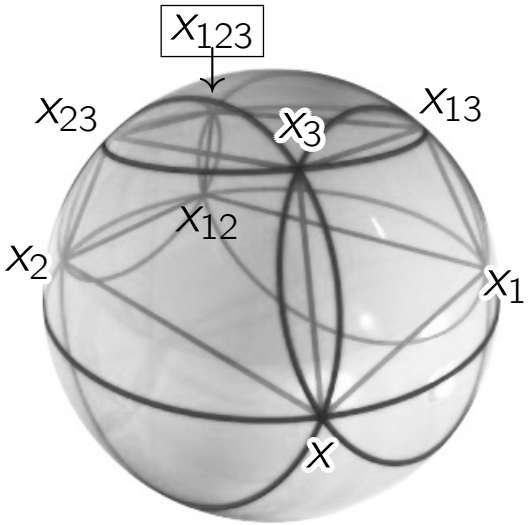


Reduction of 3-systems

- conjugate nets live in projective geometry
- circular nets live in Möbius geometry
- Oval quadric (sphere) in projective space is a model Möbius geometry — planar sections are circle, projective automorphisms of quadric are Möbius transforms
- conjugate net in the quadric is a circular net

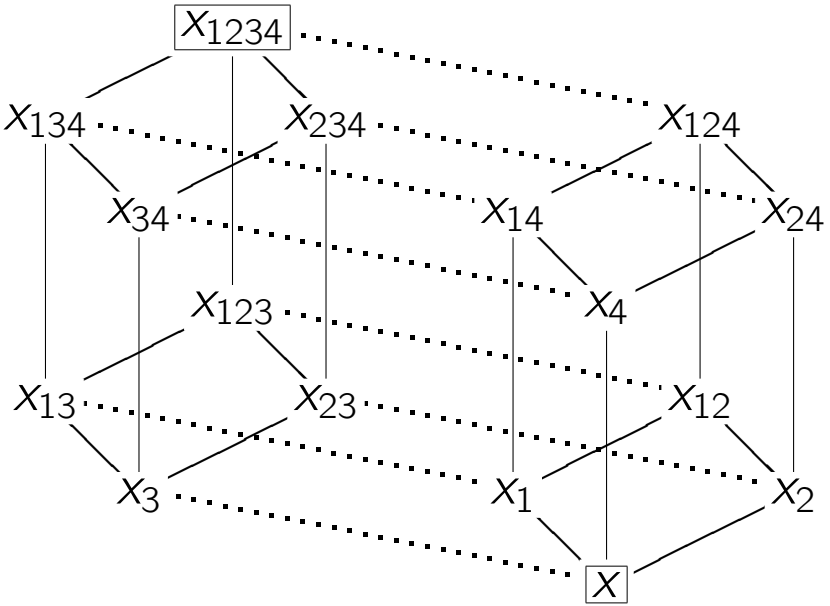
Propagation of Circularity

- Consider a conjugate net $x(u_1, \ldots, u_n)$ $(n \geq 3)$. In the generic case circularity is implied by circular initial values (i.e., quads containing vertices with $u_1 u_2 \cdots u_n = 0$ are circular).
- Proof: Consider a 3-cell with diagonal x—x_{123} where the quads incident to x are circular.
- Miquel $\Longrightarrow$ x_{123} exists is intersection of circles. Conjugacy $\Longrightarrow$ x_{123} is *already* determined as intersection of planes.



4-consistency of 3-systems

- Can we construct a conjugate net $x(u_1, u_2, u_3, u_4)$ from conjugate initial values? (quads having a vertex with $u_1 u_2 u_3 u_4 = 0$ are planar)
- Make 3-cubes incident with x conjugate: These are x—x_{123}, x—x_{124}, x—x_{134}, and x—x_{234}.
- Now there are four competing ways to find x_{1234}: one for each 3-cube incident with x_{1234}.



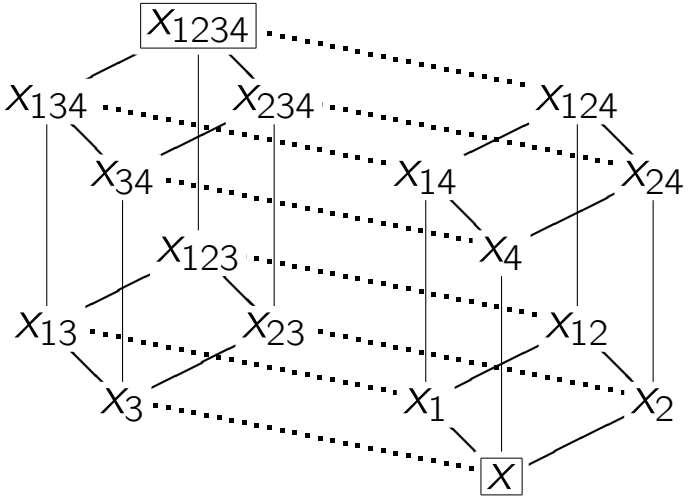
4-consistency of 3-systems

- **Thm.** Conjugacy is an integrable (i.e., 4-consistent) 3-system.
- **Proof** in dimensions ≥ 4: Within cube x_1—x_{1234}, we have

$$x_{1234} \in \mathrm{span}(x_{12}x_{123}x_{124}) \cap \mathrm{span}(x_{13}x_{132}x_{134}) \cap \mathrm{span}(x_{14}x_{142}x_{143}).$$

- Similar expressions for other cubes
- each quad is intersection of two cubes

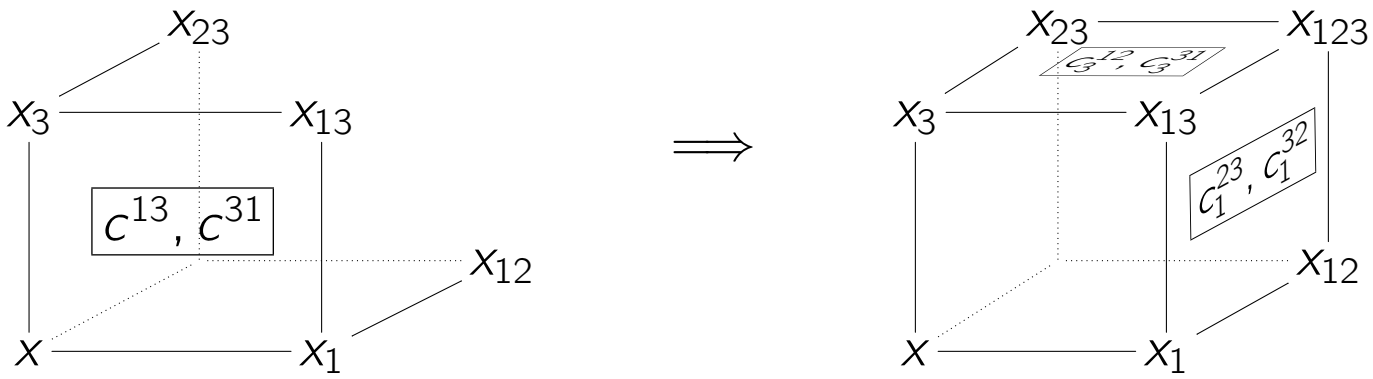
  $\Longrightarrow \; x_{1234} \in \bigcap_{i=1}^{\not 04}(\mathrm{span}(\ldots))$
- that expression no longer depends on choice of initial 3-cube, q.e.d.



4-consistency of 3-systems

- **Lemma.** Consider coefficient functions c^{lk}, c^{kl} in a conjugate 3- net defined by $\Delta_k \Delta_l x = c^{lk} \Delta_k x + c^{kl} \Delta_l x$. There is a birational mapping

$$(c^{12}, c^{21}, c^{23}, c^{32}, c^{31}, c^{13}) \xmapsto{\phi} (c_3^{12}, c_3^{21}, c_1^{23}, c_1^{32}, c_2^{31}, c_2^{13})$$



4-consistency of 3-systems

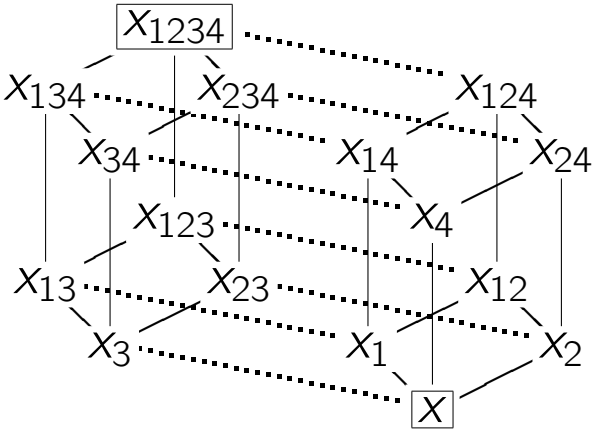
Proof. Use the "product rule" $\Delta_j(a \cdot b) = a_j \cdot \Delta_j b + \Delta_j a \cdot b$ to expand

$$\begin{aligned}
\Delta_i\Delta_j\Delta_k x &= \Delta_i(c^{kj}\Delta_j x + c^{jk}\Delta_k x) = c_i^{kj}\Delta_i\Delta_j x + \Delta_i c^{kj}\Delta_j x + \cdots \\
&= (c_i^{kj}c^{ji} + c_i^{jk}c^{ki})\Delta_i x + (c_i^{kj}c^{ij} + \Delta_i c^{kj})\Delta_j x + (c_i^{jk}c^{ik} + \Delta_i c^{jk})\Delta_k x. \\
\Delta_j\Delta_k\Delta_i x &= (c_j^{ik}c^{kj} + c_j^{ki}c^{ij})\Delta_j x + (c_j^{ik}c^{jk} + \Delta_j c^{ik})\Delta_k x + (c_j^{ki}c^{ji} + \Delta_j c^{ki})\Delta_i x. \\
\Delta_k\Delta_i\Delta_j x &= (c_k^{ji}c^{ik} + c_k^{ij}c^{jk})\Delta_k x + (c_k^{ji}c^{ki} + \Delta_k c^{ji})\Delta_i x + (c_k^{ij}c^{kj} + \Delta_k c^{ij})\Delta_j x.
\end{aligned}$$

Permutation invariance of $\Delta_i\Delta_j\Delta_k$ yields linear system for var. $(c_3^{12}, \ldots, c_1^{32})$, namely equations $\Delta_i c^{jk} = c_k^{ij}c^{jk} + c_k^{ji}c^{ik} - c_i^{jk}c^{ik}$ $(i \neq j \neq k \neq i)$ valid in the generic case. Matrix inversion $\implies$ desired birational mapping.

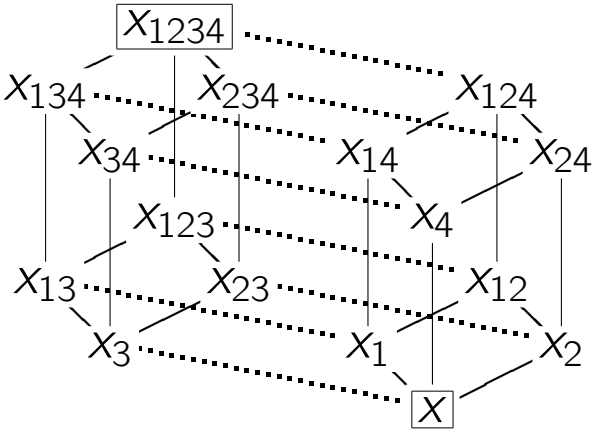
4-consistency of 3-systems

- **Thm.** Conjugacy is an integrable (i.e., 4-consistent) 3-system.
- **Proof.** general position and dimension $d \geq 4$: see above.
- Alternative: computation x_{1234} by the birational mapping
  $(c^{ik}, c^{ki}, c^{kl}, c^{lk}, c^{li}, c^{il}) \xmapsto{\phi} (c_l^{ik}, c_l^{ki}, c_i^{kl}, c_i^{lk}, c_k^{li}, c_k^{il})$
  which applies to $\Delta_k \Delta_l x = c^{lk} \Delta_k x + c^{kl} \Delta_l x$.
- Different computations yield the same result in case $d \geq 4 \implies$ identity of rational functions $\implies$ same result in *all* cases.



N-consistency of 3-systems

- **Thm.** Circularity is an integrable (i.e., 4-consistent) 3-system.
- **Proof.** Circularity propagates through a conjugate net.
- **Thm.** 4-consistency implies N-consistency for $N \geq 4$.
- **Proof.** ($N = 5$) Suppose 4-cubes x—x_{1234}, x—x_{1235}, ... x—x_{2345}. are conjugate $\implies$ each 4-cube incident with x_{12345} is conjugate, with a possible conflict regarding x_{12345}.
- There is no conflict because any two 4-cubes share a 3-cube.



3-systems: Summary

- Discrete conjugate surfaces resp. circular surfaces are discrete versions of smooth conjugate surfaces resp. principal surfaces.
- Conjugacy is a 3-system in d-dimensional projective space ($d \geq 3$), and circularity is a 3-system in d-dimensional Möbius geometry ($d \geq 2$).
- Both 3-systems are 4-consistent, i.e., integrable.
- 4-consistency implies N-consistency for all $N \geq 4$.

Part III

2-Systems

K-surfaces

- For a smooth surface $x(u_1, u_2)$ which is asymptotic, i.e.,

$$\partial_{11}x, \partial_{22}x \in \mathrm{span}\{\partial_1 x, \partial_2\},$$

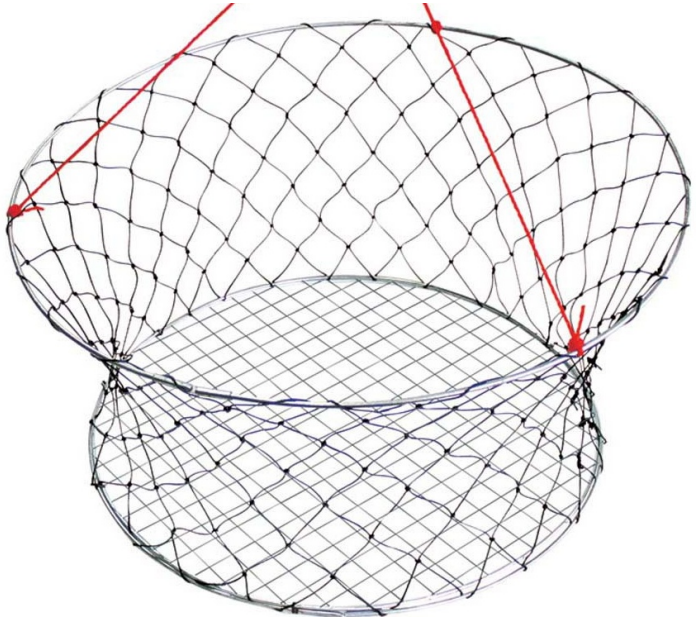
$K = \mathrm{const}$ is characterized by the Chebyshev condition

$$\partial_2\|\partial_1 x\| = \partial_1\|\partial_2 x\| = 0$$

- discrete n-dim. case:

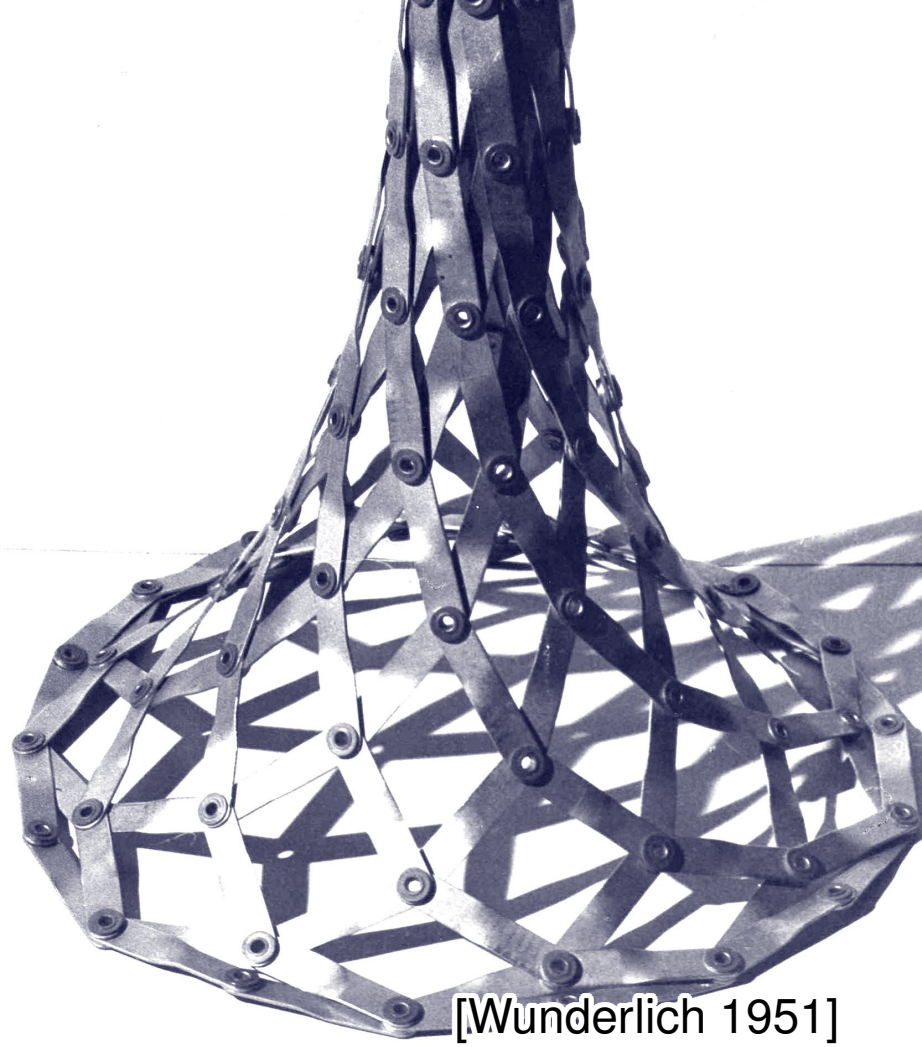
$$x, x_k, x_{\bar{k}}, x_l, x_{\bar{l}} \in \text{plane } P(u)$$

$$\Delta_l\|\Delta_k x\| = \Delta_k\|\Delta_l x\| = 0$$



K-surfaces

- unit normal vector $n(u)$,
  rotation angle $\cos\alpha_k = \langle n, n_k \rangle$
- $\Delta_l \|\Delta_k x\| = 0 \ (l \neq k)$
  $\Delta_l \langle n, n_k \rangle = 0 \ (l \neq k)$
- $x(u_1, u_2)$ is uniquely determined by initial values $x(0, u_2)$ and $x(u_1, 0)$ (2-system property)
- $\Longrightarrow$ flexible mechanism made of flat twisted members



[Wunderlich 1951]

Multidim. consistency of K-nets

- **Thm.** A discrete K-net $x(u_1, \ldots, u_d)$ has unit normal vectors n which obey $\Delta_l \langle n, n_k \rangle = 0$ $(k \neq l)$ (Chebyshev property), besides other conditions (T-net property).

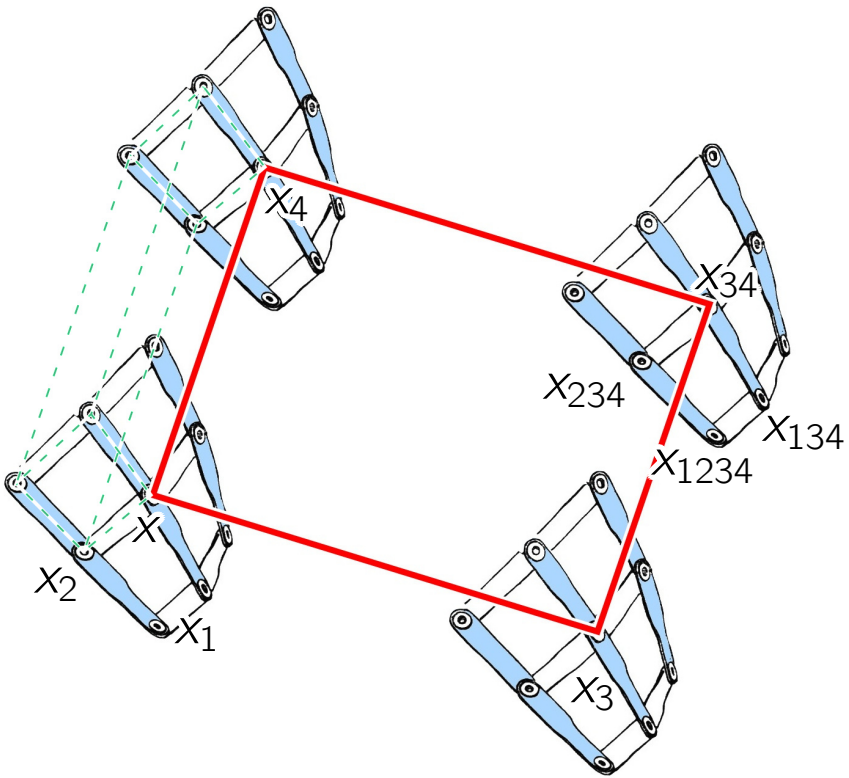
  The K-net property is d-consistent for $d \geq 3$ (i.e., integrable): any choice of initial values

  $$n(i_1, 0, \ldots, 0), \quad n(0, i_1, \ldots, 0), \quad \ldots, \quad n(0, 0, \ldots, i_d) \quad (d \geq 2)$$

  can be extended to a net of unit normal vectors, and in turn defines a K-surface.

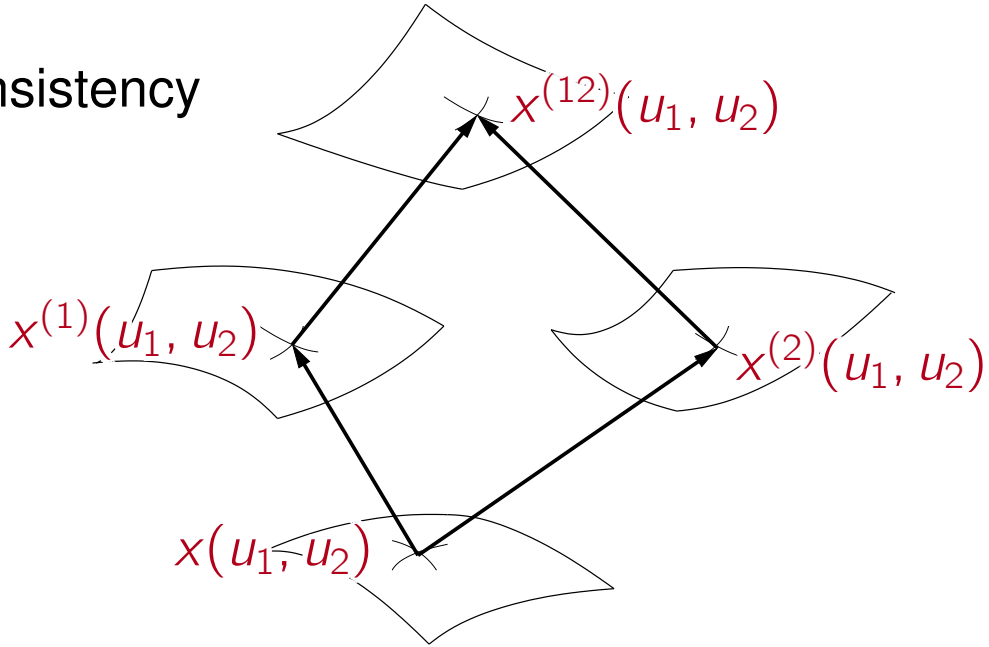
Multidim. consistency of K-nets

- A 3-dimensional K-surface consists of sequential Bäcklund transforms of 2D K-surfaces
- A 4-dimensional K-surface is a lattice of Bäcklund transforms
- (Curiosity: Bennet's 1906 4-bar and 12-bar mechanisms)



Bäcklund transform

- Existence of Bäcklund transforms for discrete K-surfaces is 3-consistency
- Bianchi permutability is 4-consistency
- This is true also for smooth K-surfaces
- Discrete theory is master theory, smooth situation obtainable by a limit



The sine-Gordon equation

- If $x(u_1, u_2)$ is smooth surface with

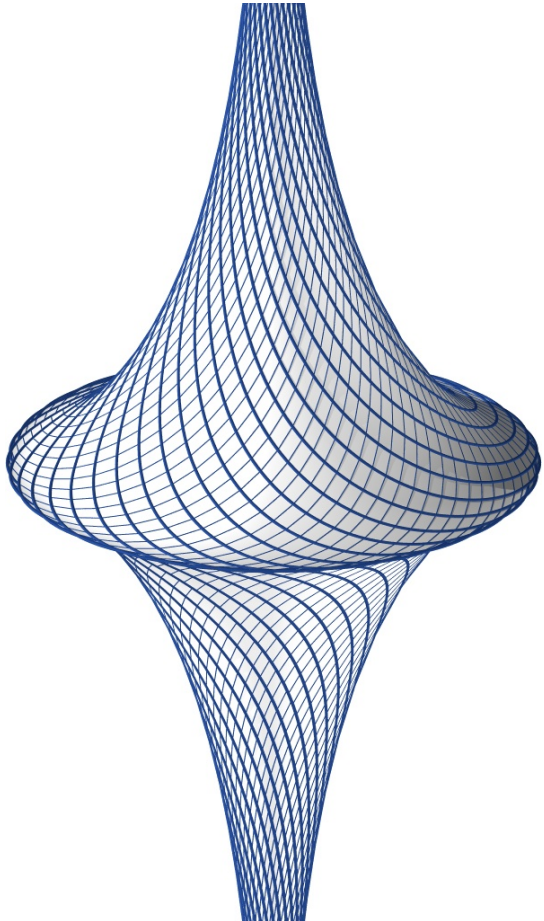
$$\|\partial_1 x\| = \|\partial_2 x\| = 1,$$

the angle between the parameter lines obeys

$$\partial_{12}\phi(u_1, u_2) = -K(u_1, u_2) \cdot \sin\phi(u_1, u_2)$$

- If $K = -1$, the angle obeys the sine-Gordon equation:

$$\partial_{12}\phi = \sin\phi.$$



The sine-Gordon equation

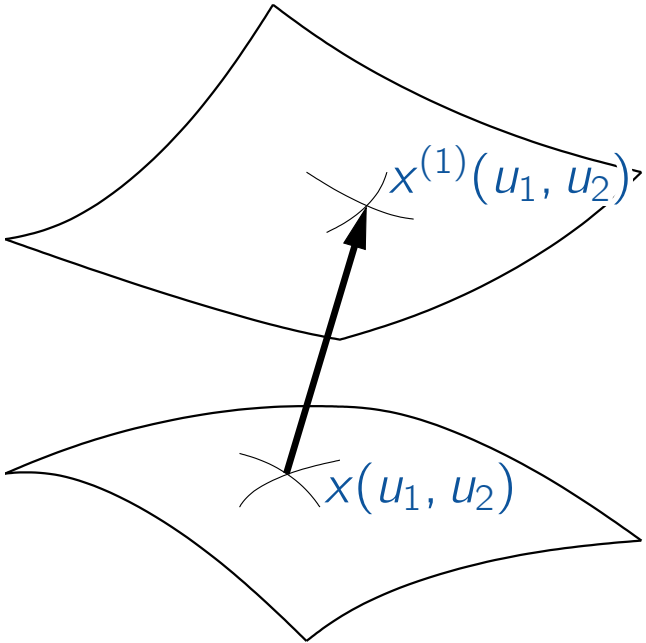
- angle ϕ between parameter lines of surface $x(u_1, u_2)$ obeys

$$\partial_{12}\phi = \sin\phi.$$

- If $x^{(1)}$ is a Bäcklund transform of x, the angle $\phi^{(1)}$ likewise obeys the sine-Gordon equation, and in addition we have

$$\partial_1\phi^{(1)} = \partial_1\phi + 2a\sin\frac{\phi + \phi^{(1)}}{2},$$

$$\partial_2\phi^{(1)} = -\partial_2\phi + \frac{2}{a}\sin\frac{\phi^{(1)} - \phi}{2}.$$



The sine-Gordon equation

- The evolution of the angle between edges in a discrete K-surface, the sine-Gordon equation and the Bäcklund transformation of the sine-Gordon equation are all consequences of the discrete Hirota equation which applies to d-dimensional discrete K-surfaces.
- Important step in the history of discrete differential geometry [Bobenko Pinkall J. Diff. Geom 1996]

Discretization Principles

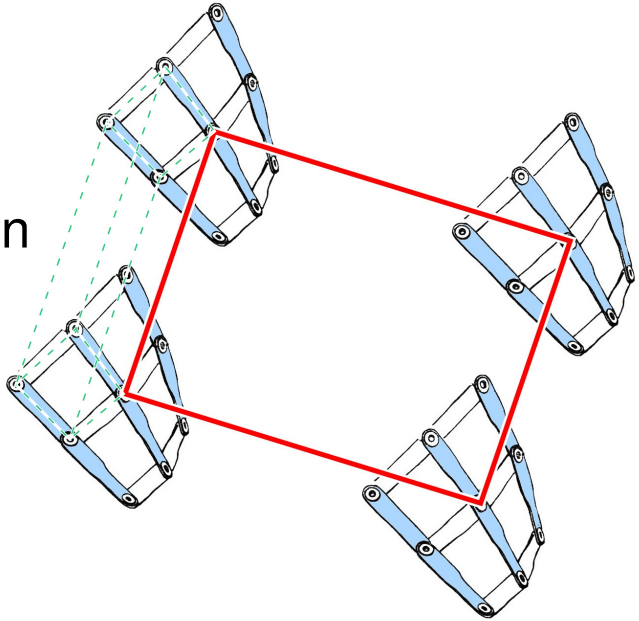
- smooth objects have several equivalent defining properties, unclear which one should be discretized
- E.g., minimal surfaces are equivalently defined . . .
  - in a variational way (local area minimization)
  - via curvatures ($H = 0$)
  - explicity, as $x(u, v) = \frac{1}{2}\,\mathrm{Re}\int_0^{u+iv} \big(f(z)(1-g(z)^2), if(z)(1+g(z)^2), 2f(z)g(z)\big)\ dz$ (Christoffel-dual transformation of a conformal parametrization of S^2)

Discretization Principles

- Good discretizations retain not only one, but several properties of the smooth object
- curious fact: the most interesting discrete versions of surfaces orignally defined by curvatures do not involve curvatures at all.
- Integrability (= consistency) is a major discretization principle

2-systems: Summary

- K-surfaces as a mechanism, as a 2-system, as a geometric incarnation of the sine-Gordon equation
- Transformations of surfaces
  $=$ higher-dimensional surfaces
- parallel devlopment: integrable discretization of surfaces, and of equations.
- Discretization principles
- Discrete theory is master theory



Part IV

Applications

Computing Minimal Surfaces

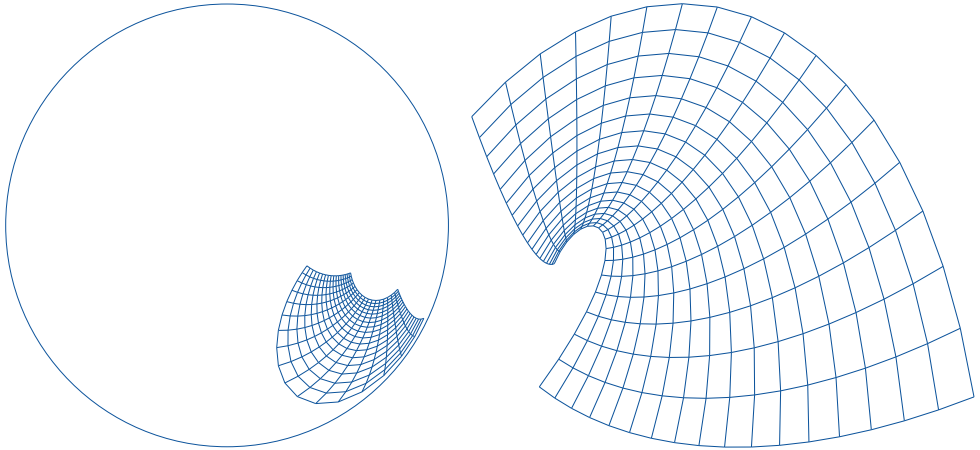
Isothermic surfaces and their duals

- **Dfn.** An isothermic surface $x(u_1, u_2)$ is a conformal principal surface, i.e., $\|\partial_1 x\| = \|\partial_2 x\|$, $\langle \partial_1 x, \partial_2 x \rangle = 0$, $\partial_{12} x \in \operatorname{span}(\partial_1 x, \partial_2 x)$.
- **Lemma.** An isothermic surface has a Christoffel-dual surface x^* (again isothermic, with $\|\partial_k x^*\| = 1/\|\partial_k x\|$) defined by

$$\partial_j x^* = (-1)^{j-1} \frac{\partial_j x}{\|\partial_k x\|^2}$$

- **Proof.** Check

$$\partial_2(\partial_1 x^*) = \partial_1(\partial_2 x^*).$$



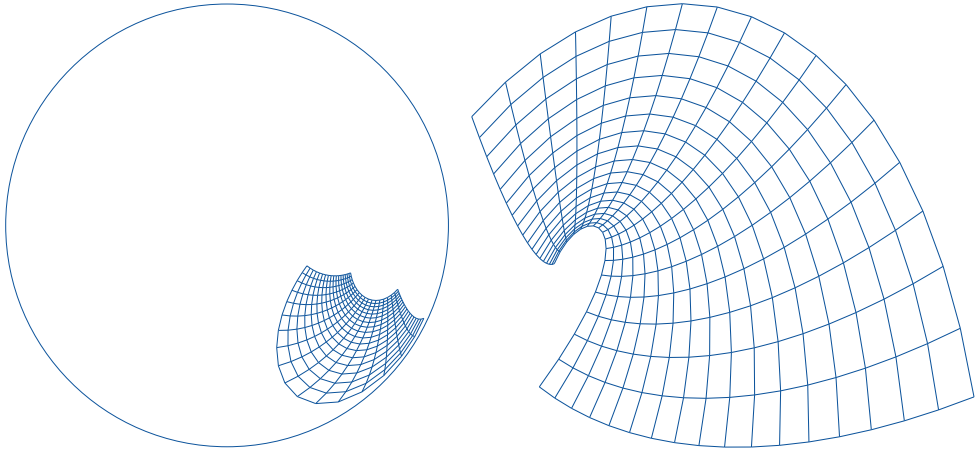
Minimal surfaces as Christoffel duals

- **Thm.** If $x(u_1, u_2)$ is a conformal parametrization of S^2, it is isothermic and its dual is a minimal surface whose normal vector field is x^*. Every minimal surface is obtained in this way.
- Some implications are easy, e.g.

$$\kappa_1 = \frac{\partial_1 x}{\partial_1 x^*} = \quad \|\partial_k x\|^2,$$

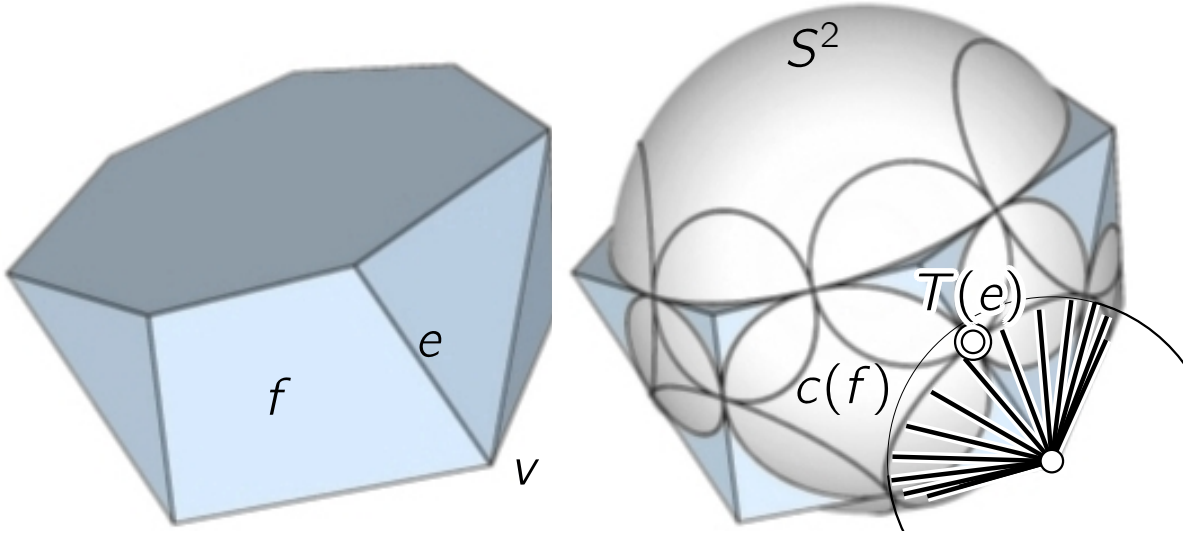
$$\kappa_2 = \frac{\partial_2 x}{\partial_2 x^*} = -\|\partial_k x\|^2$$

$$\Longrightarrow \quad H = \frac{\kappa_1 + \kappa_2}{2} = 0.$$



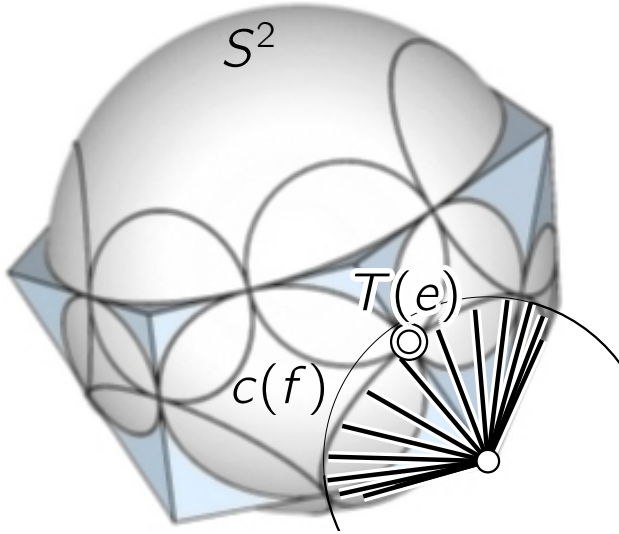
Koebe polyhedra

- **Thm.** For each convex polyhedron P there is a combinatorially equivalent convex polyhedron P' whose edges are tangent to the unit sphere (*Koebe polyhedron*).
- P' is unique up to Möbius transforms, exactly one P' has its center of mass in 0.



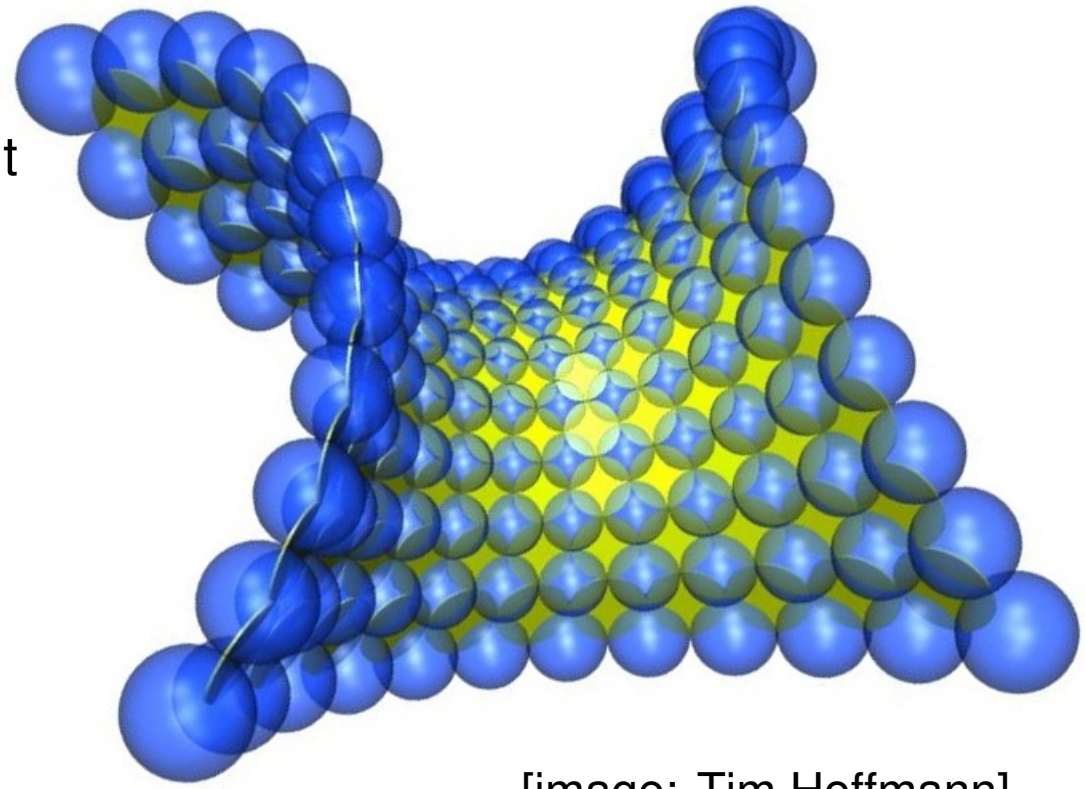
Discrete s-isothermic surfaces

- **Dfn.** S-isothermic surfaces are polyhedra where faces $f \in F$ have incircles $c(f)$, vertices $v \in V$ are centers of spheres $S(v)$, and edges $e \in E$ carry points $T(e)$ s.t.
  - $v \in e \implies S(v)$ intersects e orthogonally in the point $T(e)$
  - $e \subset f \implies c(f)$ touches e in $T(e)$
  - $\forall v \in V, \deg(v) = 4$. $\forall f \in F,\ \deg(f)$ even
- Koebe polyhedra are s-isothermic, if ...



Discrete s-isothermic surfaces

- s-isothermic surface is circle-sphere arrangement
- combinatorially regular parts of s-isothermic surfaces (resp. Koebe polyhedra) are regarded as conformal parametrizations

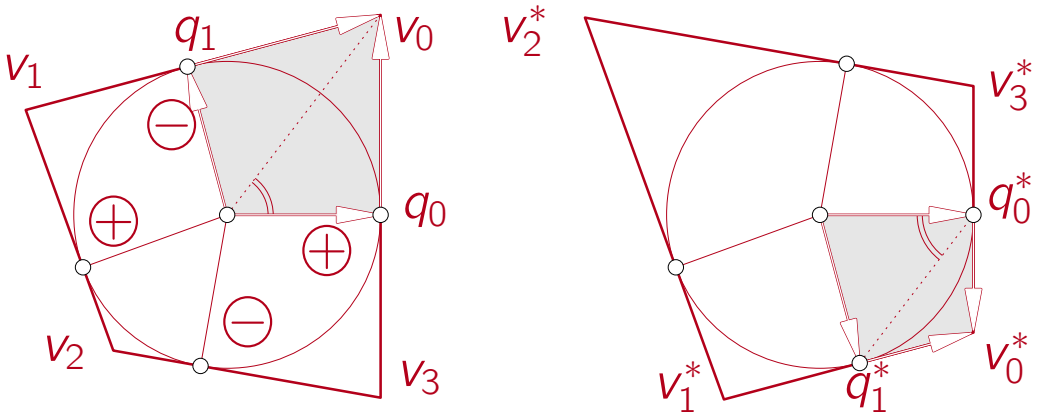


[image: Tim Hoffmann]

Dualizing polygons

- **Lemma.** A $2n$-gon with incircle has a dual as follows:

radius ρ



$$q_i^* = \bigoplus_i \frac{q_i}{\rho^2}$$

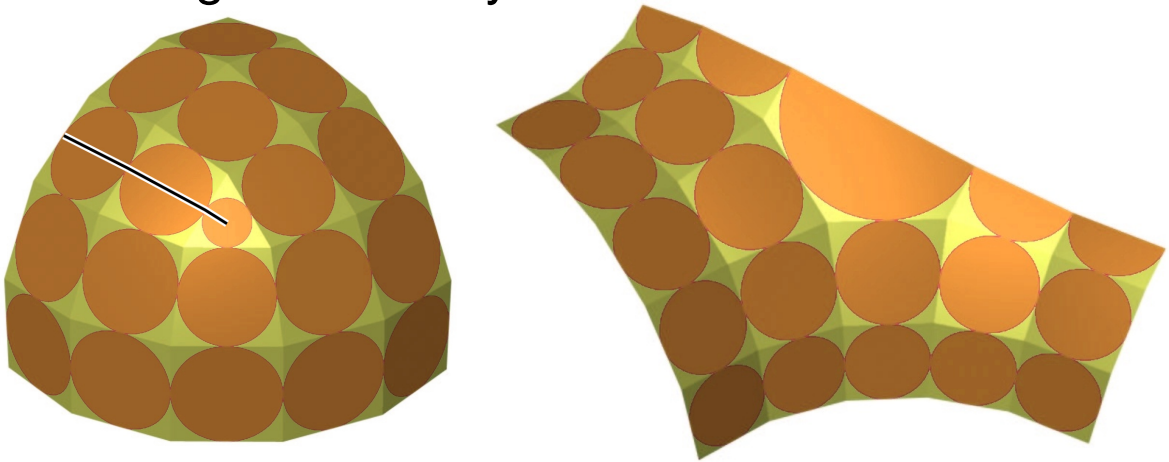
radius $1/\rho$

- **Lemma.** Subdivision into quads yields discrete surfaces with

$$\Delta x_1^* = \frac{1}{\|\Delta x_1\|^2}\Delta x_1, \quad \Delta x_2^* = -\frac{1}{\|\Delta x_2\|^2}\Delta x_2.$$

Christoffel duality of surfaces

- **Prop.** An s-isothermic surface has a *Christoffel dual* surface, s.t. corresponding faces are dual in the above sense.
- **Proof.** Labels $\oplus$, $\ominus$ can be consistently assigned. By elementary geometry, dual lenghts and angles fit locally.

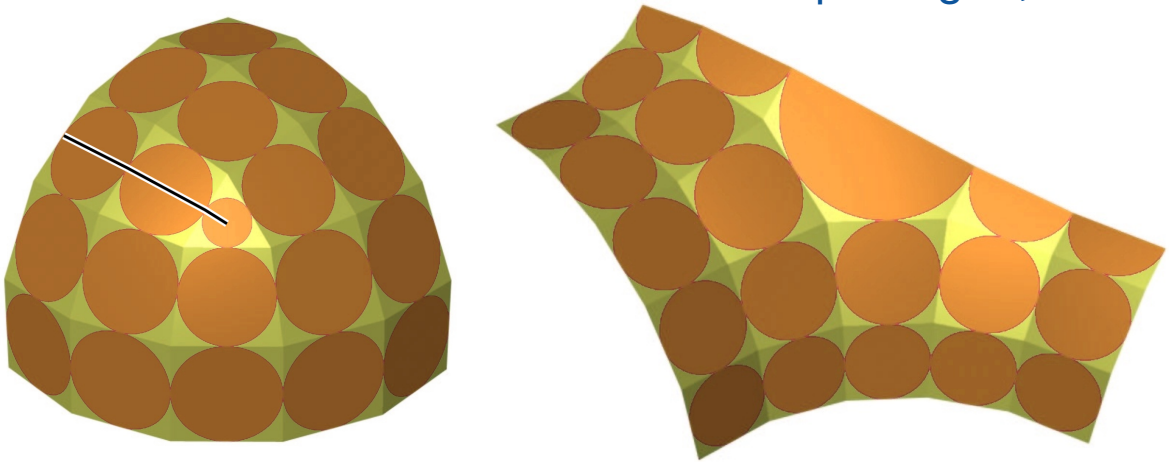


[images: B. Springborn]

Discrete minimal surfaces

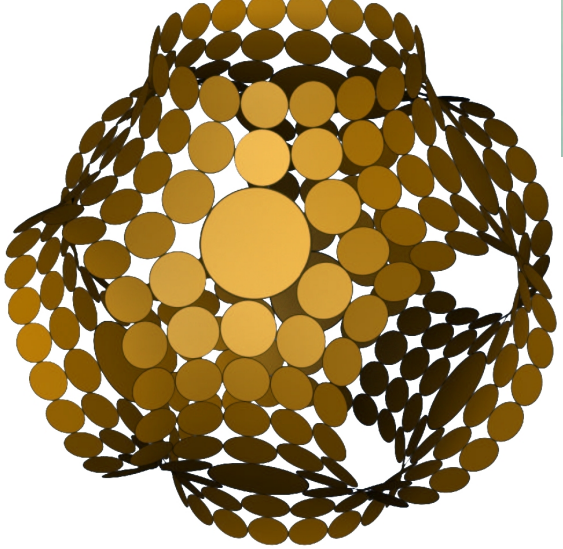
- **Dfn.** Minimal surface is dual of Koebe s-isothermic surface
- **Thm.** Convergence to principal curves of minimal surface
- **Proof.** via convergence of circle packings to conformal mappings

[O. Schramm. Circle patterns with the combinatorics of the square grid, Duke Math. J. 1997]



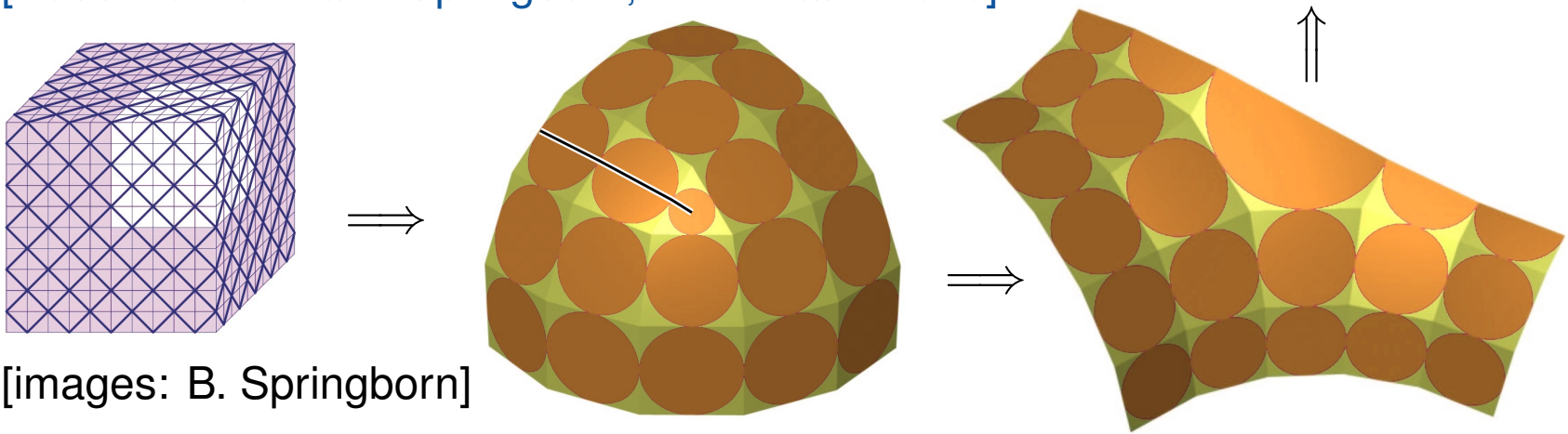
[images: B. Springborn]

Computing minimal surf.



- minimal surface $\Rightarrow$ principal curves $\Rightarrow$ map to unit sphere $\Rightarrow$ cell decomposition of sphere $\Rightarrow$ Koebe polyhedron $\Rightarrow$ discrete minimal surface

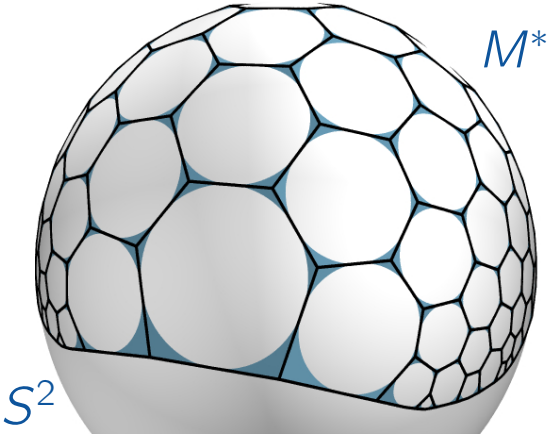
  [Bobenko Hoffmann Springborn, Ann. Math. 2006]

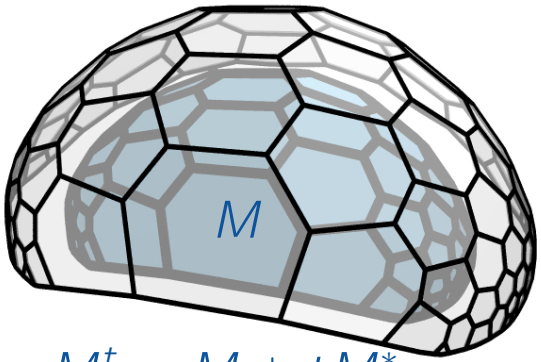


[images: B. Springborn]

Constant-distance offset surfaces

- Constant-speed evolution $x^t = x + t \cdot n(x)$ of a smooth surface
- Constant-speed evolution $v^t = v + t \cdot v^*$ of a polyhedral surface M, guided by a combinatorially equivalent surface M^* whose edges/faces are parallel to those of M.
- linear space of polyhedral surfaces parallel to M
- $M^* \approx S^2$





$M^t = M + tM^*$

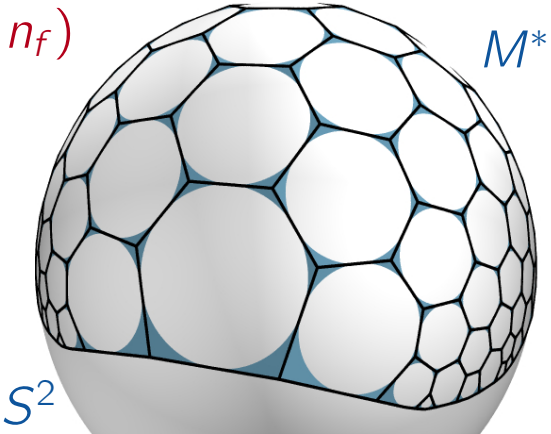
Curvatures of discrete surfaces

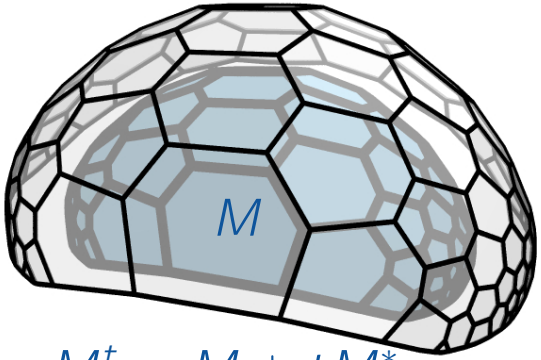
- Evolution of smooth surface: $dA^t(x) = (1 - 2H(x)t + K(x)t^2)\, dA(x)$.
- Evolution of discrete surface: $A(f^t) = A(f) + 2tA(f, f^*) + t^2 A(f^*)$

$$A(f^t) = A(f)\left(1 - 2t\frac{A(f, f^*)}{A(f)} + t^2\frac{A(f^*)}{A(f)}\right)$$

- $A(f) = \frac{1}{2}\sum_{i=0}^{n-1} \det(v_i, v_{i+1}, n_f)$
- $H(f) = -\frac{A(f, f^*)}{A(f)}$

  $K(f) = \frac{A(f^*)}{A(f)}$





$M^t = M + tM^*$

Curvatures of discrete surfaces

- By means of areas and mixed areas, a mean curvature and Gauss curvature can be assigned to the individual faces of a polyhedral surface M^*, if M is endowed with an appropriate Gauss image M^*.
- **Lemma** S-isothermic minimal surfaces enjoy $H = 0$.
- classes of discrete surfaces whose continuous originals are defined by curvatures, now can be equipped with curvatures too.

  [Bobenko Pottmann W Math. Ann 2010]

  [Hoffmann, Sageman-Furnas, Wardetzky IMRN 2015]

Summary

- The duals of Koebe polyhedra are discrete minimal surfaces
- This can be used to compute the shape of minimal surfaces form the combinatorics of their network of principal curvature lines
- Despite being constructed by means of other discretization principles, classes of discrete surfaces can be endowed with curvatures

Part V

Applications: Freeform Architecture

List of topics

- steel-glass constructions following polyhedral surfaces / single-curved glass — conjugate surfaces
- torsion-free support structures / self-supporting surfaces / material-minimizing forms — curvatures of discrete surfaces
- regular patterns — circle packings, discrete conformal mapping
- paneling free forms — assignment problems with optimization
- the design dilemma — numerical methdods like energy-guided projection

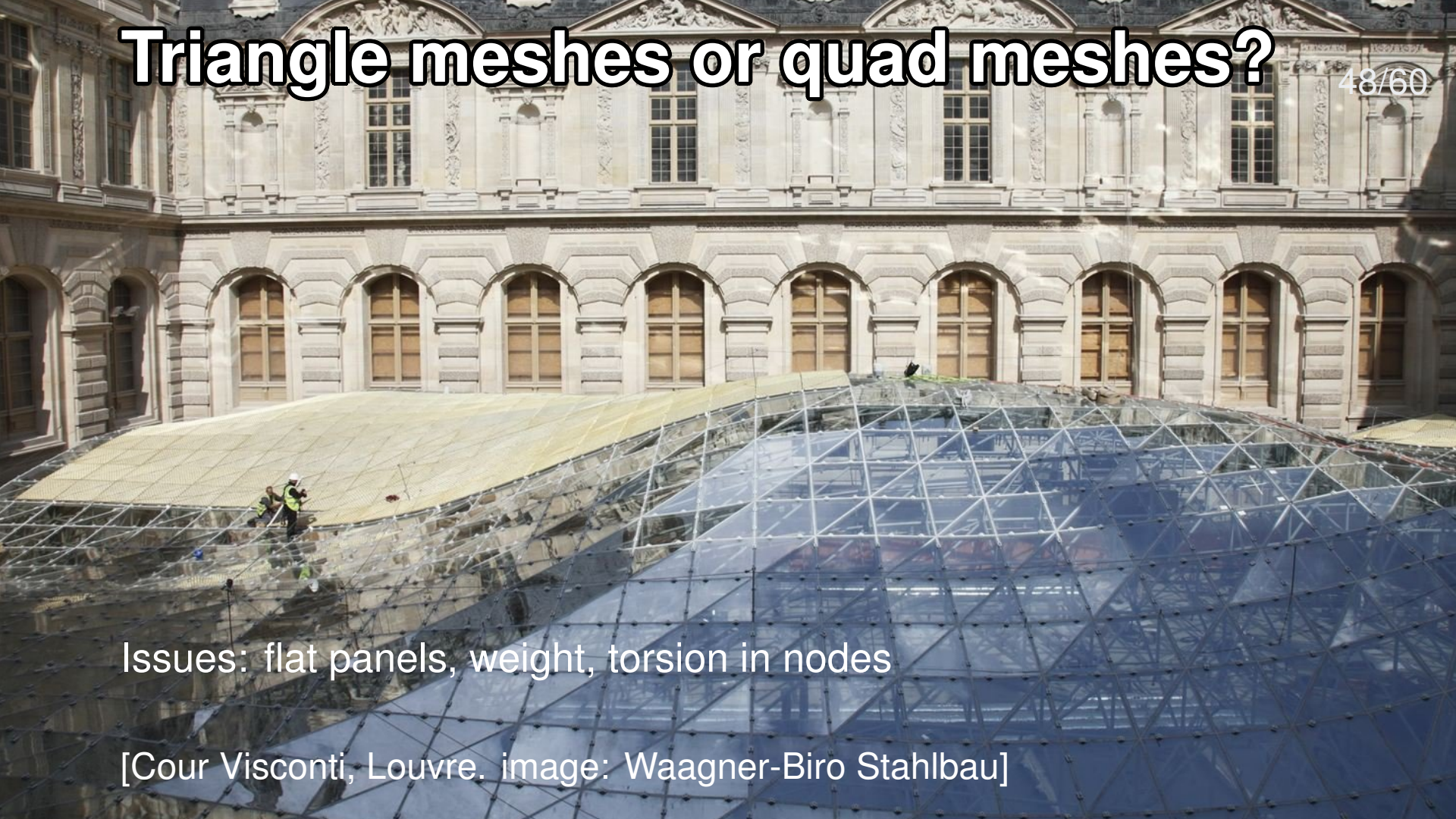
What are free forms?

- Some structures are easy to design (at least, as an amorphous surface), but cannot be fully designed easily or built cheaply
- Special cases are easier, but are no true free forms

St. Lazare, Paris

hippo house, Berlin zoo

Triangle meshes or quad meshes?

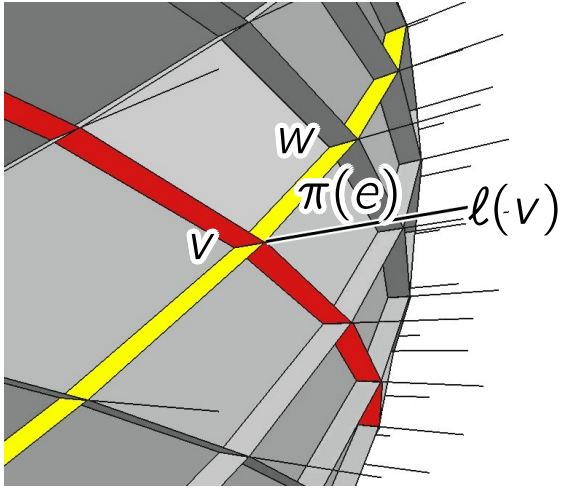


Issues: flat panels, weight, torsion in nodes

[Cour Visconti, Louvre. image: Waagner-Biro Stahlbau]

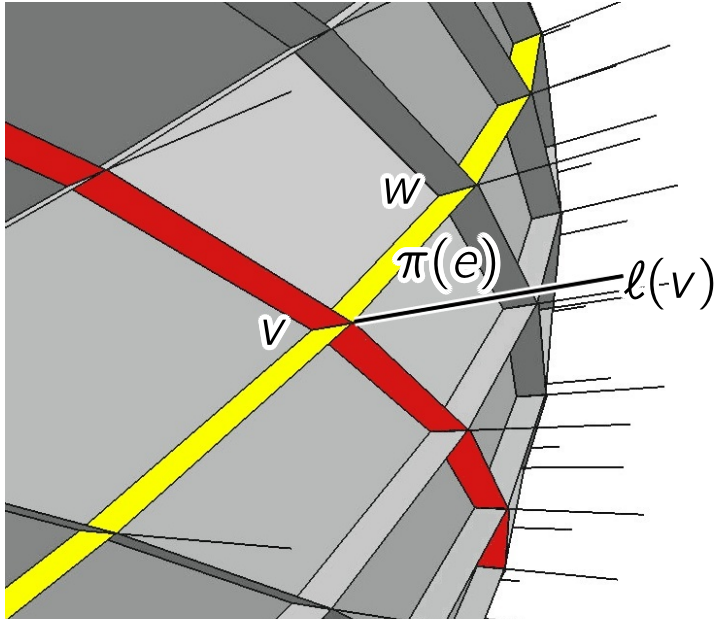
Torsion-free support structures

- **Def.** A torsion-free support structure associated to a mesh (V, E, F) edges is an assignment of a line $\ell(v)$ to each vertex v and a plane $\pi(e)$ to each edge e, such that $v \in e \implies \ell(v) \subset \pi(e)$



Torsion-free support structures

- align straight beams with planes $\pi(e)$
- clean ($\Longrightarrow$ cheaply built) intersection

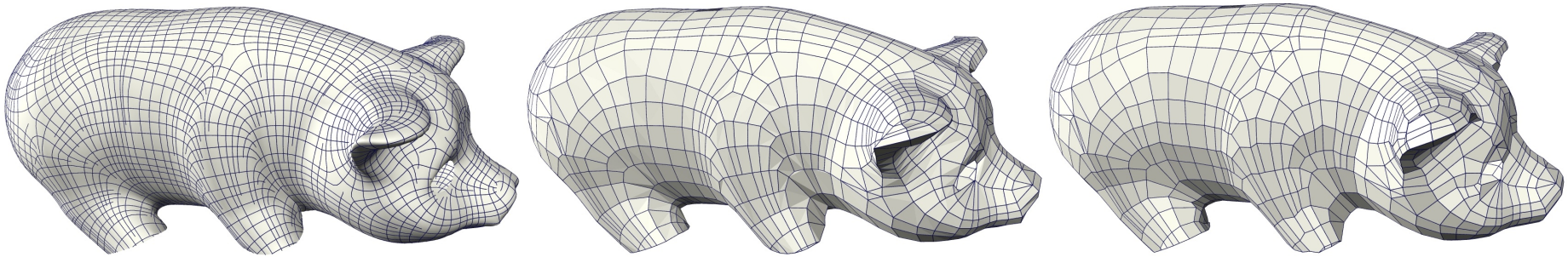


Torsion-free support structures

- **Prop.** A triangle mesh has only trivial support structures, where all elements pass through a single center z.
- **Proof** For each face $f = v_i v_j v_k$, we have $\ell(v_i) = \pi(v_i v_j) \cap \pi(v_i v_k)$, and similar for $\ell(v_j)$, $\ell(v_k)$.
- $z(f) = \pi(v_i v_j) \cap \pi(v_i v_k) \cap \pi(v_j v_k)$ lies on all lines $\ell(v_i)$, $\ell(v_j)$, $\ell(v_k)$.
- If $f' = (v_i v_j v_l)$ is a neighbor face, then $z(f) = \ell(v_i) \cap \ell(v_j) = z(f')$.
- Quad meshes are “better” than triangle meshes, as far as complexity of nodes is concerned are concerned

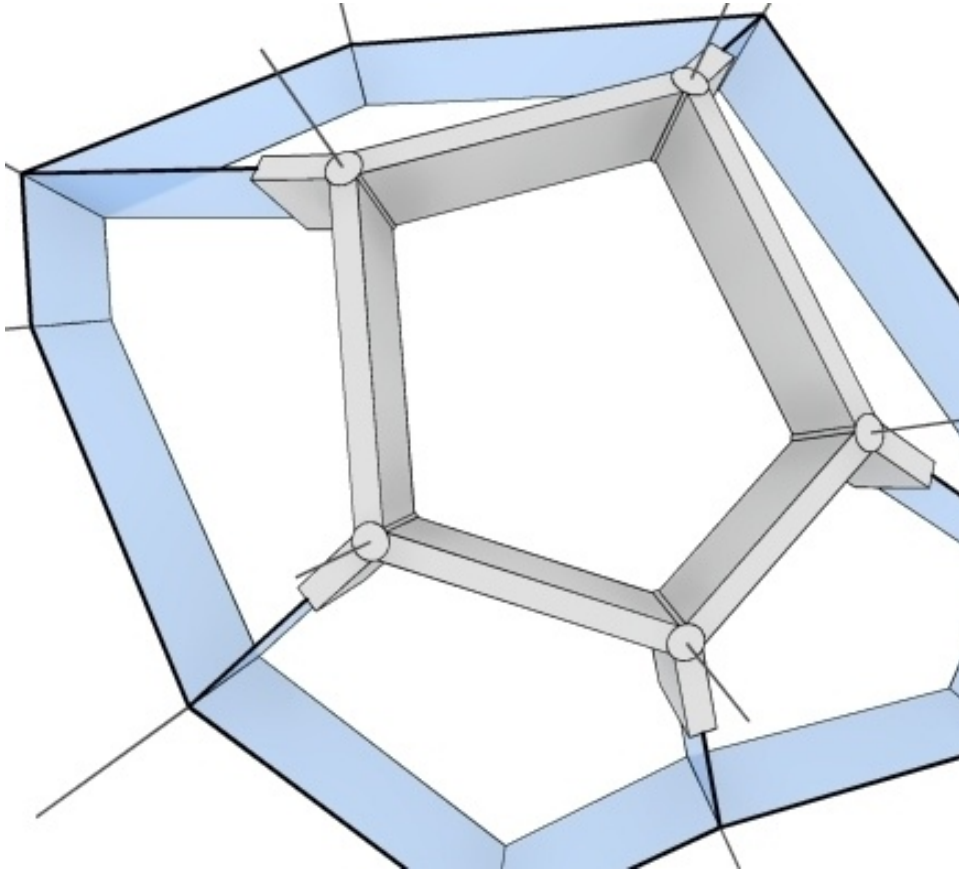
Quad meshes with planar faces

- start of architecture applications of DDG [Liu et al, SIGGRAPH 2006]
- Cannot simply “optimize” a quad mesh for conjugacy (planar faces). Geometric shape determines net to great extent
- Only recently, interactive modelling has become efficient enough for the planarity constraint [Tang et al, SIGGRAPH 2014]



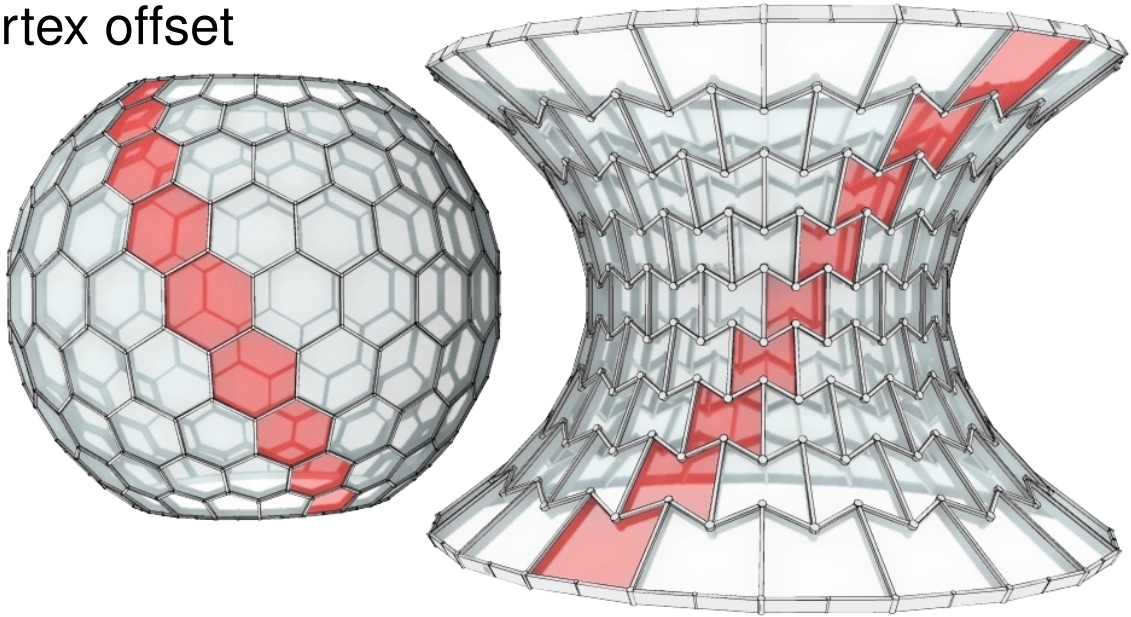
Parallel meshes

- Meshes with planar faces are parallel, if corresponding edges and faces are
- Parallel meshes define a torsion-free support structure, and vice versa.



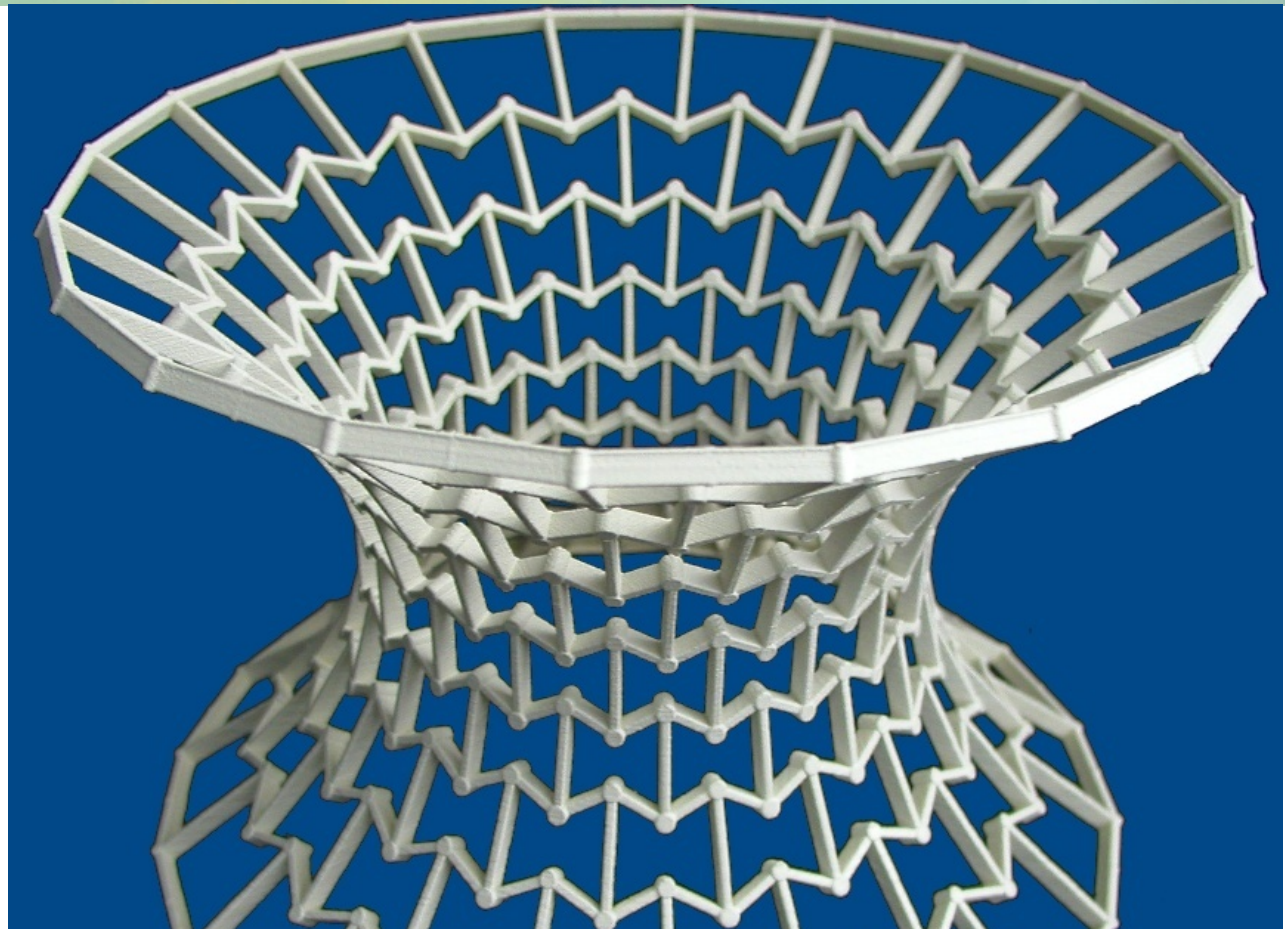
Constant-distance offsets

- If M, M^* are parallel with $M^* \approx S^2$, vertex-wise linear combination $M^t = M + tM^*$ yields an offset of M at constant distance t
- M^* inscribed in S^2 — vertex offset
- M^* circumscribed — face offset
- M^* midscribed (Koebe) — edge offset
- existence: 3-system



Constant-distance offsets

- application: multilayer constructions
- here: beams of constant height



The design dilemma

- If Mathematics is involved in the design phase, definite solutions of problems are unwelcome.
- Unique solutions restrict freedom of artistic expression
- Example: Eiffel Tower pavilions

The design dilemma

- Eiffel tower pavilions
- curved beams are developable & orthogonal to glass surfaces
- $\Longrightarrow$ they follow principal curvature lines; their layout is defined already by the glass surface
- Solution: impreceptibly change glass until principal curves fit

Freeform architecture: Summary

- DDG occurs in the realization of free forms
- Design dilemma
- Goal for the future: geometry-aware computational design.

Conclusion

- Multidimensional consistency as a discretization principle
- Examples of 2-systems and 3-systems
- Applications within Mathematics
- Applications outside Mathematics

Discretization in
Geometry and Dynamics
SFB Transregio 109

Der Wissenschaftsfonds.