Discrete Parametric Surfaces

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Short Course on Discrete Differential Geometry
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Overview

- Introduction and Notation
- An integrable 3-system: circular surfaces
- An integrable 2-system: K-surfaces
- Computing minimal surfaces
- Freeform architecture
Part I

Introduction and Notation
Parametric Surfaces

- Surfaces $x(u)$ where $u \in \mathbb{Z}^k$ or $u \in \mathbb{R}^k$ or $u \in \mathbb{Z}^k \times \mathbb{R}^l$

- continuous case, discrete case, mixed case
- Surfaces $x(u)$ where $u \in \mathbb{Z}^k$ or $u \in \mathbb{R}^k$ or $u \in \mathbb{Z}^k \times \mathbb{R}^l$
- continuous case, discrete case, mixed case
- Transformations of $(k - 1)$-dimensional surfaces yield $k$-dim. surfaces
Examples of discrete surfaces

- quad meshes in freeform architecture

[Yas Marina Hotel, Abu Dhabi] [Hippo house, Berlin zoo]
Examples of discrete surfaces

- various mappings
  \[ \mathbb{Z}^k \rightarrow \{ \text{points} \} \text{ or } \mathbb{Z}^k \rightarrow \{ \text{spheres} \} \text{ or } \ldots \]
- Discrete minimal surfaces,
  discrete cmc surfaces,
  discrete K-surfaces,
  etc.

[images: Tim Hoffmann]
Notation

- discrete parametric surface $\cong$ net
- Right shift of a surface $x$ in the $k$-th direction is denoted by $x_k$
- Left shift of a surface $x$ in the $k$-th direction is denoted $x_{\bar{k}}$

\[
\begin{align*}
  x_1(k, l) &= x(k + 1, l) \\
x_2(k, l) &= x(k, l + 1)
\end{align*}
\]

- Differences are used instead of derivatives $\Delta_i x = x_i - x$
Surface classes: conjugate surfaces

- a smooth conjugate surface $x(u)$ has “planar” infinitesimal quads

$$3 \text{ vol } \left( \text{c.h. } (x(u), x(u_1+\varepsilon, u_2), x(u_1, u_2+\varepsilon), x(u_1+\varepsilon, u_2+\varepsilon)) \right)/\varepsilon^4$$

$$= \det(x(u_1+\varepsilon, u_2) - x(u), x(u_1, u_2+\varepsilon) - x(u), x(u_1+\varepsilon, u_2+\varepsilon) - x(u)) \frac{1}{2\varepsilon^4}$$

$$\approx \det(\varepsilon \partial_1 x, \varepsilon \partial_2 x, \varepsilon \partial_1 x + \varepsilon \partial_2 x + 2\varepsilon^2 \partial_{12} x)/2\varepsilon^4$$

$$= \det(\partial_1 x, \partial_2, \partial_{12} x) = 0$$
Surface classes: conjugate surfaces

- A discrete conjugate surface $x(u)$ has planar elementary quads.
- $\det(\Delta_k x, \Delta_l x, \Delta_{kl} x) = 0 \iff x_{kl}$ is co-planar for all $l, k \iff$ (generically)
- There are coefficient functions $c^{lk}, c^{kl}$ s.t. $\Delta_k \Delta_l x = c^{lk} \Delta_k x + c^{kl} \Delta_l x$. 
Parameter lines are asymptotic, i.e., in every point they indicate the intersection of the saddle-shaped surface with its own tangent plane.

**smooth case:** \( \partial_{11}x, \partial_{22}x \in \text{span}\{\partial_1x, \partial_2\} \).

**discrete case:**

\[
x, x_1, x_\bar{1}, x_2, x_\bar{2} \in \text{plane } P(u)
\]
Surface classes: principal surfaces

- smooth case: parameter lines are conjugate + orthogonal
  \[ x_l - x_{kl} \]
- discrete case: \[ x - x_k \] is circular for all \( l, k \)
Why is circularity “principal”?

- smooth case: principal curves are characterized by developability of the surface formed by normals
- discrete case: normals are circles’ axes. Observe developability
- (convergence can be proved)
- [A. Bobenko, Yu. Suris: Discrete Differential Geometry, AMS 2009]
Part II

3-systems
Conjugacy as a 3-system

- Generically, a conjugate net $x(u_1, u_2, u_3)$ is uniquely determined by arbitrary initial values $x(0, u_2, u_3)$ and $x(u_1, 0, u_3)$ and $x(u_1, u_2, 0)$.
- Proof is trivial (intersect planes)
Generically, a circular net $x(u_1, u_2, u_3)$ is uniquely determined by arbitrary initial values $x(0, u_2, u_3)$ and $x(u_1, 0, u_3)$ and $x(u_1, u_2, 0)$.

Proof is not trivial (Miquel’s theorem)
Reduction of 3-systems

- conjugate nets live in projective geometry
- circular nets live in Möbius geometry
- Oval quadric (sphere) in projective space is a model Möbius geometry — planar sections are circle, projective automorphisms of quadric are Möbius transforms
- conjugate net in the quadric is a circular net
Consider a conjugate net \( x(u_1, \ldots, u_n) (n \geq 3) \). In the generic case circularity is implied by circular initial values (i.e., quads containing vertices with \( u_1 u_2 \cdots u_n = 0 \) are circular).

Proof: Consider a 3-cell with diagonal \( x \rightarrow x_{123} \) where the quads incident to \( x \) are circular.

Miquel \( \rightarrow \) \( x_{123} \) exists is intersection of circles. Conjugacy \( \rightarrow \) \( x_{123} \) is already determined as intersection of planes.
Can we construct a conjugate net \( x(u_1, u_2, u_3, u_4) \) from conjugate initial values? (quads having a vertex with \( u_1 u_2 u_3 u_4 = 0 \) are planar)

Make 3-cubes incident with \( x \) conjugate: These are \( x \rightarrow x_{123} \), \( x \rightarrow x_{124} \), \( x \rightarrow x_{134} \), and \( x \rightarrow x_{234} \).

Now there are four competing ways to find \( x_{1234} \): one for each 3-cube incident with \( x_{1234} \).
Thm. Conjugacy is an integrable (i.e., 4-consistent) 3-system.

Proof in dimensions ≥ 4: Within cube \( x_1 \cdots x_{1234} \), we have

\[
x_{1234} \in \text{span}(x_{12}x_{123}x_{124}) \cap \text{span}(x_{13}x_{132}x_{134}) \cap \text{span}(x_{14}x_{142}x_{143}).
\]

Similar expressions for other cubes

each quad is intersection of two cubes

\[
\Rightarrow x_{1234} \in \bigcap_{i=1}^{64} (\text{span}(\ldots))
\]

that expression no longer depends on choice of initial 3-cube, q.e.d.
Lemma. Consider coefficient functions $c^{lk}, c^{kl}$ in a conjugate 3-net defined by $\Delta_k \Delta_l x = c^{lk} \Delta_k x + c^{kl} \Delta_l x$. There is a birational mapping

$$(c^{12}, c^{21}, c^{23}, c^{32}, c^{31}, c^{13}) \mapsto (c_3^{12}, c_3^{21}, c_1^{23}, c_1^{32}, c_2^{31}, c_2^{13})$$
Proof. Use the “product rule” $\Delta_j(a \cdot b) = a_j \cdot \Delta_j b + \Delta_j a \cdot b$ to expand

$$\Delta_i \Delta_j \Delta_k x = \Delta_i(c^{kj} \Delta_j x + c^{jk} \Delta_k x) = c^{kj}_i \Delta_i \Delta_j x + \Delta_i c^{kj} \Delta_j x + \cdots$$

$$= (c^{kj}_i c^{ji} + c^{ij}_i c^{ki}) \Delta_i x + (c^{kj}_i c^{ij} + \Delta_i c^{kj}) \Delta_j x + (c^{jk}_i c^{ik} + \Delta_i c^{jk}) \Delta_k x.$$

$$\Delta_j \Delta_k \Delta_i x = (c^{ik}_j c^{kj} + c^{ki}_j c^{ij}) \Delta_j x + (c^{ik}_j c^{jk} + \Delta_j c^{ik}) \Delta_k x + (c^{kj}_i c^{ji} + \Delta_j c^{kj}) \Delta_i x.$$

$$\Delta_k \Delta_i \Delta_j x = (c^{ji}_k c^{ik} + c^{ij}_k c^{jk}) \Delta_k x + (c^{ji}_k c^{ki} + \Delta_k c^{ji}) \Delta_i x + (c^{ij}_k c^{kj} + \Delta_k c^{ij}) \Delta_j x.$$

Permutation invariance of $\Delta_i \Delta_j \Delta_k$ yields linear system for var. $(c^{12}_3, \ldots, c^{32}_1)$, namely equations $\Delta_i c^{jk} = c^{ij}_k c^{jk} + c^{ij}_k c^{ik} - c^{jk}_i c^{ik} (i \neq j \neq k \neq i)$ valid in the generic case. Matrix inversion $\implies$ desired birational mapping.
Thm. Conjugacy is an integrable (i.e., 4-consistent) 3-system.

Proof. general position and dimension $d \geq 4$: see above.

Alternative: computation $x_{1234}$ by the birational mapping

$$(c^{ik}, c^{ki}, c^{kl}, c^{li}, c^{il}) \mapsto (c^{ik}_i, c^{ki}_i, c^{kl}_i, c^{li}_i, c^{il}_i)$$

which applies to $\Delta_k \Delta_l x = c^{lk}_k \Delta_k x + c^{kl}_l \Delta_l x$.

Different computations yield the same result in case $d \geq 4 \implies$ identity of rational functions $\implies$ same result in all cases.
Thm. Circularity is an integrable (i.e., 4-consistent) 3-system.

Proof. Circularity propagates through a conjugate net.

Thm. 4-consistency implies $N$-consistency for $N \geq 4$.

Proof. ($N = 5$) Suppose 4-cubes $x \rightarrow x_{1234}$, $x \rightarrow x_{1235}$, ... $x \rightarrow x_{2345}$. are conjugate $\implies$ each 4-cube incident with $x_{12345}$ is conjugate, with a possible conflict regarding $x_{12345}$.

There is no conflict because any two 4-cubes share a 3-cube.
Discrete conjugate surfaces resp. circular surfaces are discrete versions of smooth conjugate surfaces resp. principal surfaces.

Conjugacy is a 3-system in $d$-dimensional projective space ($d \geq 3$), and circularity is a 3-system in $d$-dimensional Möbius geometry ($d \geq 2$).

Both 3-systems are 4-consistent, i.e., integrable.

4-consistency implies $N$-consistency for all $N \geq 4$. 
Part III

2-Systems
K-surfaces

- For a smooth surface $x(u_1, u_2)$ which is asymptotic, i.e.,

$$\partial_{11}x, \partial_{22}x \in \text{span}\{\partial_1x, \partial_2\},$$

$K = \text{const}$ is characterized by the Chebyshev condition

$$\partial_2\|\partial_1x\| = \partial_1\|\partial_2x\| = 0$$

- discrete $n$-dim. case:

$x, x_k, x_i, x_j \in \text{plane } P(u)$

$$\Delta_i\|\Delta_kx\| = \Delta_k\|\Delta_ix\| = 0$$
K-surfaces

- Unit normal vector $n(u)$,
- Rotation angle $\cos \alpha_k = \langle n, n_k \rangle$
- $\Delta_l \| \Delta_k x \| = 0 \ (l \neq k)$
- $\Delta_l \langle n, n_k \rangle = 0 \ (l \neq k)$
- $x(u_1, u_2)$ is uniquely determined by initial values $x(0, u_2)$ and $x(u_1, 0)$ (2-system property)
- $\implies$ Flexible mechanism made of flat twisted members
Thm. A discrete K-net $x(u_1, \ldots, u_d)$ has unit normal vectors $n$ which obey $\Delta_l \langle n, n_k \rangle = 0$ ($k \neq l$) (Chebyshev property), besides other conditions (T-net property).

The K-net property is $d$-consistent for $d \geq 3$ (i.e., integrable): any choice of initial values

$$n(i_1, 0, \ldots, 0), \quad n(0, i_1, \ldots, 0), \quad \ldots, \quad n(0, 0, \ldots, i_d) \quad (d \geq 2)$$

can be extended to a net of unit normal vectors, and in turn defines a K-surface.
Multidim. consistency of K-nets

- A 3-dimensional K-surface consists of sequential Bäcklund transforms of 2D K-surfaces
- A 4-dimensional K-surface is a lattice of Bäcklund transforms
- (Curiosity: Bennet’s 1906 4-bar and 12-bar mechanisms)
- Existence of Bäcklund transforms for discrete K-surfaces is 3-consistency
- Bianchi permutability is 4-consistency
- This is true also for smooth K-surfaces
- Discrete theory is master theory, smooth situation obtainable by a limit
The sine-Gordon equation

- If \( x(u_1, u_2) \) is smooth surface with
  \[ \| \partial_1 x \| = \| \partial_2 x \| = 1, \]
  the angle between the parameter lines obeys
  \[ \partial_{12} \phi(u_1, u_2) = -K(u_1, u_2) \cdot \sin \phi(u_1, u_2) \]

- If \( K = -1 \), the angle obeys the sine-Gordon equation:
  \[ \partial_{12} \phi = \sin \phi. \]
The sine-Gordon equation

- angle $\phi$ between parameter lines of surface $x(u_1, u_2)$ obeys
  \[
  \partial_{12}\phi = \sin \phi.
  \]

- If $x^{(1)}$ is a Bäcklund transform of $x$, the angle $\phi^{(1)}$ likewise obeys the sine-Gordon equation, and in addition we have
  \[
  \begin{align*}
  \partial_1\phi^{(1)} &= \partial_1\phi + 2a\sin\frac{\phi + \phi^{(1)}}{2}, \\
  \partial_2\phi^{(1)} &= -\partial_2\phi + \frac{2}{a}\sin\frac{\phi^{(1)} - \phi}{2}.
  \end{align*}
  \]
The sine-Gordon equation

- The evolution of the angle between edges in a discrete K-surface, the sine-Gordon equation and the Bäcklund transformation of the sine-Gordon equation are all consequences of the discrete Hirota equation which applies to $d$-dimensional discrete K-surfaces.
- Important step in the history of discrete differential geometry

[Bobenko Pinkall J. Diff. Geom 1996]
smooth objects have several equivalent defining properties, unclear which one should be discretized

E.g., minimal surfaces are equivalently defined . . .

- in a variational way (local area minimization)
- via curvatures ($H = 0$)
- explicitly, as $x(u, v) = \frac{1}{2} \text{Re} \int_0^{u+i} \left( f(z)(1-g(z)^2), i f(z)(1+g(z)^2), 2f(z)g(z) \right) dz$ (Christoffel-dual transformation of a conformal parametrization of $S^2$)
Good discretizations retain not only one, but several properties of the smooth object.

- curious fact: the most interesting discrete versions of surfaces originally defined by curvatures do not involve curvatures at all.
- Integrability (\(=\) consistency) is a major discretization principle.
K-surfaces as a mechanism, as a 2-system, as a geometric incarnation of the sine-Gordon equation

Transformations of surfaces
  = higher-dimensional surfaces

parallel development: integrable discretization of surfaces, and of equations.

Discretization principles

Discrete theory is master theory
Part IV

Applications

Computing Minimal Surfaces
**Dfn.** An isothermic surface $x(u_1, u_2)$ is a conformal principal surface, i.e., $\|\partial_1 x\| = \|\partial_2 x\|$, $\langle \partial_1 x, \partial_2 x \rangle = 0$, $\partial_{12} x \in \text{span}(\partial_1 x, \partial_2 x)$.

**Lemma.** An isothermic surface has a Christoffel-dual surface $x^*$ (again isothermic, with $\|\partial_k x^*\| = 1/\|\partial_k x\|$) defined by

$$\partial_j x^* = (-1)^{j-1} \frac{\partial_j x}{\|\partial_k x\|^2}$$

**Proof.** Check

$$\partial_2(\partial_1 x^*) = \partial_1(\partial_2 x^*).$$
Thm. If \( \chi(u_1, u_2) \) is a conformal parametrization of \( S^2 \), it is isothermic and its dual is a minimal surface whose normal vector field is \( \chi^* \). Every minimal surface is obtained in this way.

Some implications are easy, e.g.

\[
\kappa_1 = \frac{\partial_1 x}{\partial_1 x^*} = \| \partial_k x \|^2,
\]

\[
\kappa_2 = \frac{\partial_2 x}{\partial_2 x^*} = -\| \partial_k x \|^2
\]

\[\implies H = \frac{\kappa_1 + \kappa_2}{2} = 0.\]
Koebe polyhedra

- **Thm.** For each convex polyhedron $P$ there is a combinatorially equivalent convex polyhedron $P'$ whose edges are tangent to the unit sphere (*Koebe polyhedron*).

- $P'$ is unique up to Möbius transforms, exactly one $P'$ has its center of mass in 0.
**Dfn.** S-isothermic surfaces are polyhedra where faces $f \in F$ have incircles $c(f)$, vertices $v \in V$ are centers of spheres $S(v)$, and edges $e \in E$ carry points $T(e)$ s.t.

- $v \in e \implies S(v)$ intersects $e$ orthogonally in the point $T(e)$
- $e \subset f \implies c(f)$ touches $e$ in $T(e)$
- $\forall v \in V, \deg(v) = 4$. $\forall f \in F, \deg(f)$ even
- Koebe polyhedra are s-isothermic, if ...
Discrete s-isothermic surfaces

- s-isothermic surface is circle-sphere arrangement
- combinatorially regular parts of s-isothermic surfaces (resp. Koebe polyhedra) are regarded as conformal parametrizations
Lemma. A $2n$-gon with incircle has a dual as follows:

- $q_0 = -\frac{\pi}{2}\rho v_0$,
- $q_1 = \frac{\pi}{2}\rho v_1$,
- $q_2 = \frac{\pi}{2}\rho v_2$,
- $q_3 = -\frac{\pi}{2}\rho v_3$.

Lemma. Subdivision into quads yields discrete surfaces with

$$\Delta x^*_1 = \frac{1}{\|\Delta x_1\|^2} \Delta x_1, \quad \Delta x^*_2 = -\frac{1}{\|\Delta x_2\|^2} \Delta x_2.$$
Prop. An s-isothermic surface has a *Christoffel dual* surface, s.t. corresponding faces are dual in the above sense.

Proof. Labels $\oplus$, $\ominus$ can be consistently assigned. By elementary geometry, dual lengths and angles fit locally.

[images: B. Springborn]
Discrete minimal surfaces

- **Dfn.** Minimal surface is dual of Koebe s-isothermic surface
- **Thm.** Convergence to principal curves of minimal surface
- **Proof.** via convergence of circle packings to conformal mappings


[images: B. Springborn]
Computing minimal surf.

- minimal surface $\Rightarrow$ principal curves $\Rightarrow$ map to unit sphere $\Rightarrow$ cell decomposition of sphere $\Rightarrow$ Koebe polyhedron $\Rightarrow$ discrete minimal surface


[images: B. Springborn]
Constant-distance offset surfaces

- Constant-speed evolution $x^t = x + t \cdot n(x)$ of a smooth surface
- Constant-speed evolution $v^t = v + t \cdot v^*$ of a polyhedral surface $M$, guided by a combinatorially equivalent surface $M^*$ whose edges/faces are parallel to those of $M$.

- Linear space of polyhedral surfaces parallel to $M$
- $M^* \approx S^2$
Curvatures of discrete surfaces

- **Evolution of smooth surface:** \( dA^t(x) = (1 - 2H(x)t + K(x)t^2) \, dA(x) \).
- **Evolution of discrete surface:** 
  \[
  A(f^t) = A(f) + 2tA(f, f^*) + t^2A(f^*)
  \]
  \[
  A(f^t) = A(f) \left( 1 - 2t \frac{A(f, f^*)}{A(f)} + t^2 \frac{A(f^*)}{A(f)} \right)
  \]

- \( A(f) = \frac{1}{2} \sum_{i=0}^{n-1} \det(v_i, v_{i+1}, n_f) \)
- \( H(f) = -\frac{A(f, f^*)}{A(f)} \)
- \( K(f) = \frac{A(f^*)}{A(f)} \)

\[ M^t = M + tM^* \]
By means of areas and mixed areas, a mean curvature and Gauss curvature can be assigned to the individual faces of a polyhedral surface $M^*$, if $M$ is endowed with an appropriate Gauss image $M^*$.

**Lemma** S-isothermic minimal surfaces enjoy $H = 0$.

classes of discrete surfaces whose continuous originals are defined by curvatures, now can be equipped with curvatures too.

[Hoffmann, Sageman-Furnas, Wardetzky IMRN 2015]
The duals of Koebe polyhedra are discrete minimal surfaces. This can be used to compute the shape of minimal surfaces from the combinatorics of their network of principal curvature lines. Despite being constructed by means of other discretization principles, classes of discrete surfaces can be endowed with curvatures.
Part V

Applications: Freeform Architecture
List of topics

- steel-glass constructions following polyhedral surfaces / single-curved glass — conjugate surfaces
- torsion-free support structures / self-supporting surfaces / material-minimizing forms — curvatures of discrete surfaces
- regular patterns — circle packings, discrete conformal mapping
- paneling free forms — assignment problems with optimization
- the design dilemma — numerical methods like energy-guided projection
What are free forms?

- Some structures are easy to design (at least, as an amorphous surface), but cannot be fully designed easily or built cheaply.

- Special cases are easier, but are no true free forms.
Triangle meshes or quad meshes?

Issues: flat panels, weight, torsion in nodes

[Cour Visconti, Louvre. image: Waagner-Biro Stahlbau]
Def. A torsion-free support structure associated to a mesh $(V, E, F)$ edges is an assignment of a line $\ell(v)$ to each vertex $v$ and a plane $\pi(e)$ to each edge $e$, such that $v \in e \implies \ell(v) \subset \pi(e)$
- align straight beams with planes $\pi(e)$
- clean (⇒ cheaply built) intersection
Prop. A triangle mesh has only trivial support structures, where all elements pass through a single center $z$.

Proof For each face $f = v_i v_j v_k$, we have $\ell(v_i) = \pi(v_i v_j) \cap \pi(v_i v_k)$, and similar for $\ell(v_j), \ell(v_k)$.

$z(f) = \pi(v_i v_j) \cap \pi(v_i v_k) \cap \pi(v_j v_k)$ lies on all lines $\ell(v_i), \ell(v_j), \ell(v_k)$.

If $f' = (v_i v_j v_l)$ is a neighbor face, then $z(f) = \ell(v_i) \cap \ell(v_j) = z(f')$.

Quad meshes are “better” than triangle meshes, as far as complexity of nodes is concerned.
Quad meshes with planar faces

- Start of architecture applications of DDG [Liu et al, SIGGRAPH 2006]
- Cannot simply “optimize” a quad mesh for conjugacy (planar faces). Geometric shape determines net to great extent
- Only recently, interactive modelling has become efficient enough for the planarity constraint [Tang et al, SIGGRAPH 2014]
Parallel meshes

- Meshes with planar faces are parallel, if corresponding edges and faces are.
- Parallel meshes define a torsion-free support structure, and vice versa.
If $M, M^*$ are parallel with $M^* \approx S^2$, vertex-wise linear combination $M^t = M + tM^*$ yields an offset of $M$ at constant distance $t$.

- $M^*$ inscribed in $S^2$ — vertex offset
- $M^*$ circumscribed
  — face offset
- $M^*$ midscribed (Koebe)
  — edge offset
- existence: 3-system
Constant-distance offsets

- application: multilayer constructions
- here: beams of constant height
The design dilemma

- If Mathematics is involved in the design phase, definite solutions of problems are unwelcome.
- Unique solutions restrict freedom of artistic expression
- Example: Eiffel Tower pavilions
The design dilemma

- Eiffel tower pavilions
- curved beams are developable & orthogonal to glass surfaces
- they follow principal curvature lines; their layout is defined already by the glass surface
- Solution: impreceptibly change glass until principal curves fit
Freeform architecture: Summary

- DDG occurs in the realization of free forms
- Design dilemma
- Goal for the future: geometry-aware computational design.
Conclusion

- Multidimensional consistency as a discretization principle
- Examples of 2-systems and 3-systems
- Applications within Mathematics
- Applications outside Mathematics