Discrete Mappings

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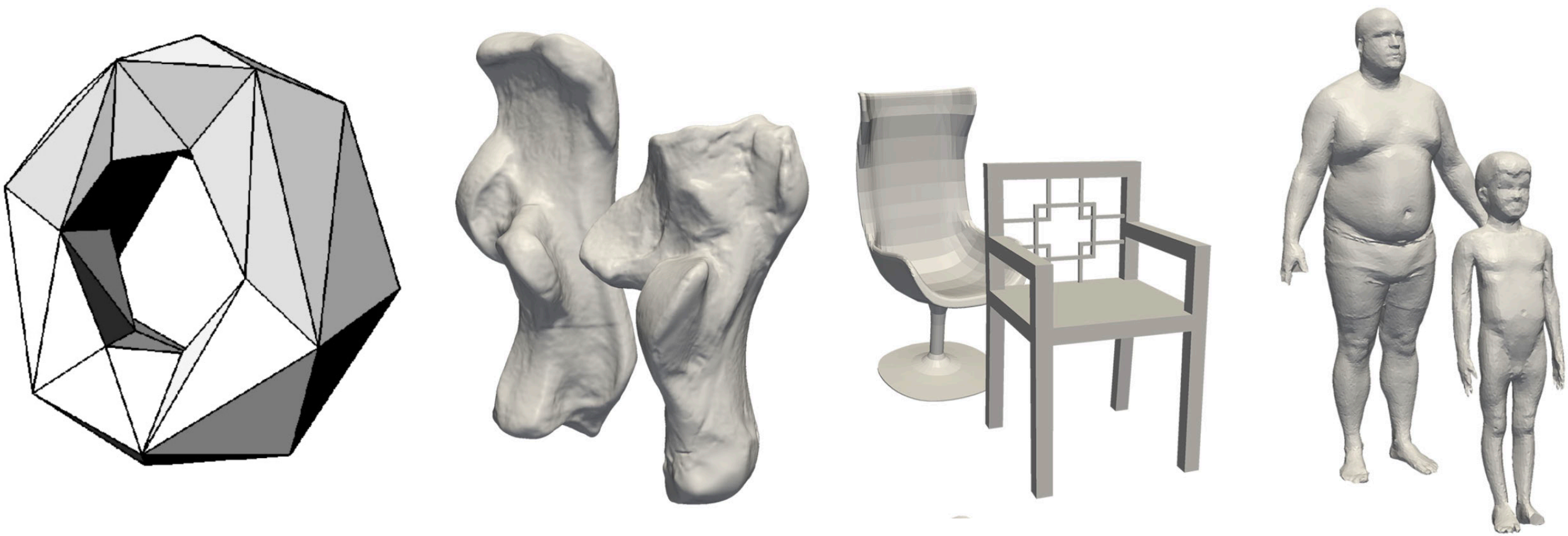
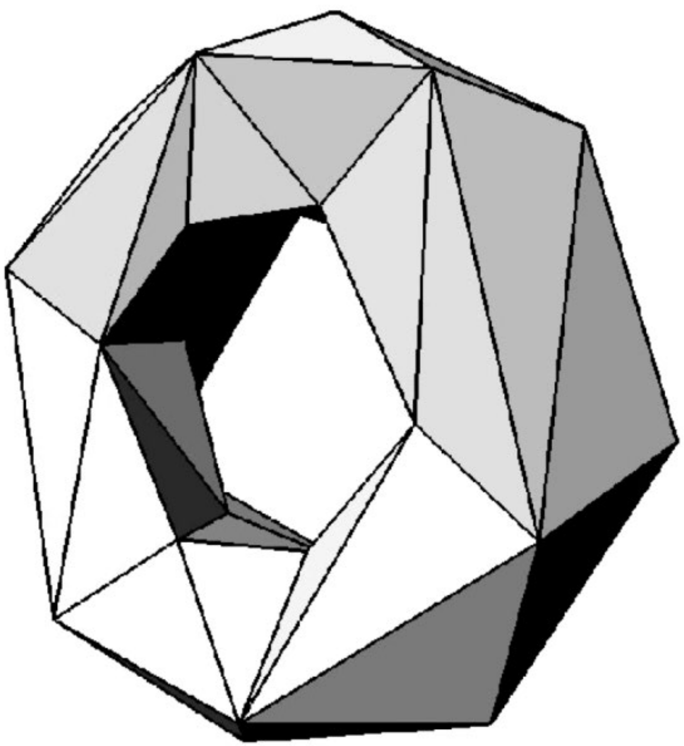
Surfaces as triangulations

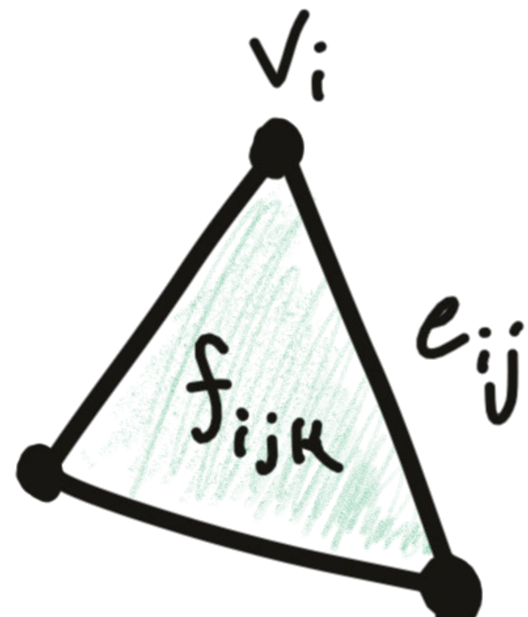
- Triangles stitched to build a surface.



Surfaces as triangulations

- Triangles stitched to build a surface.
- $M = (V, E, F)$;
- $V = \{v_i\}, E = \{e_{ij}\}, F = \{f_{ijk}\}$.

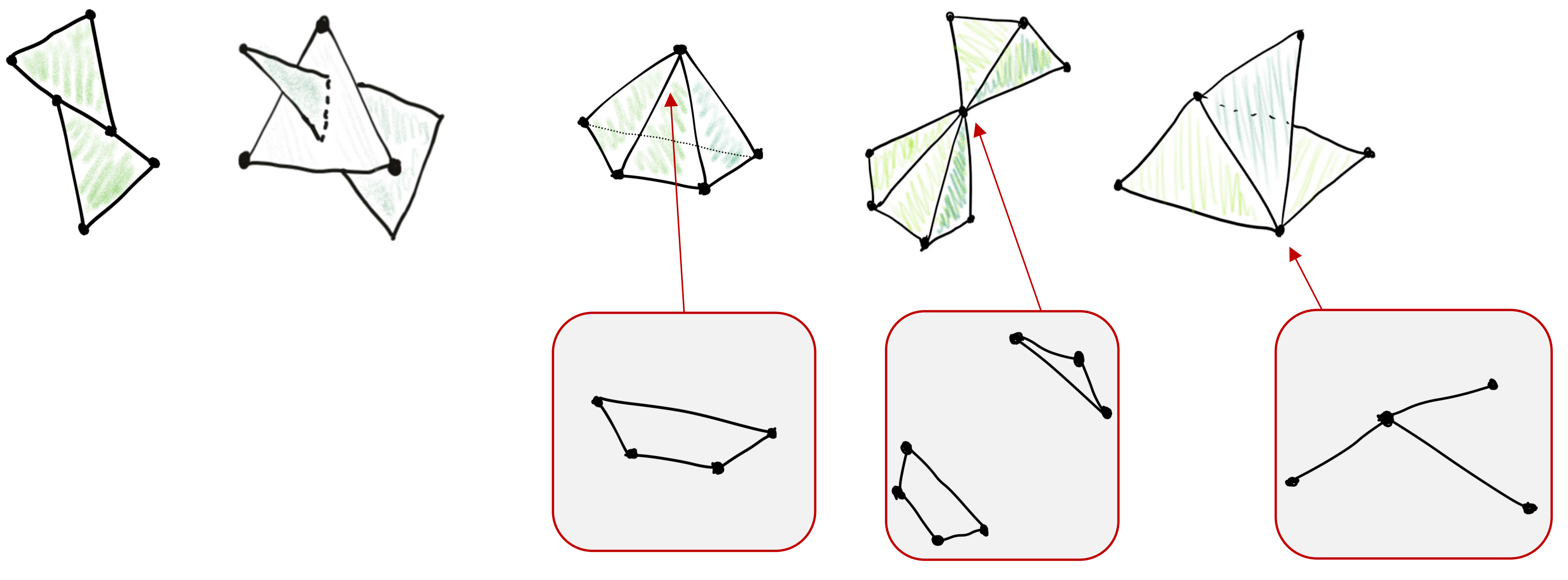




Surfaces as triangulations

- $M = (V, E, F)$

- Rules for stitching triangles:
  1. M is a simplicial complex.
  2. $\text{link}(v_i) = \cup_{\{f_{ijk} \in F\}} e_{jk}$ is a simple closed polygon.

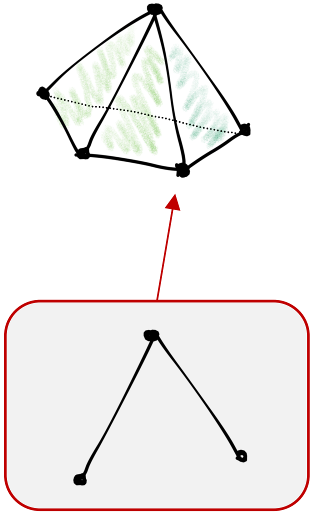


Surfaces as triangulations

- Definition. A **surface triangulation** is a triplet $M = (V, E, F)$ satisfying the stitching rules.

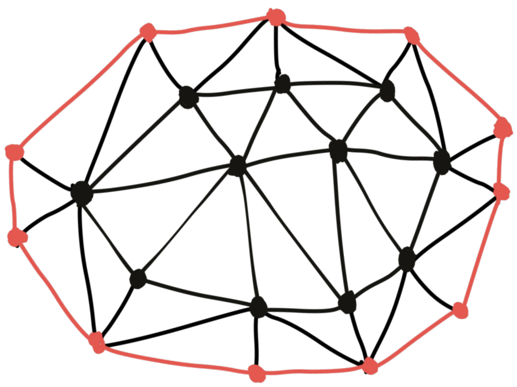
- For surface triangulation with boundary, replace stitching rule 2 with

  2. $\mathrm{link}(v_i)$ is a simple (closed or not) polygon.



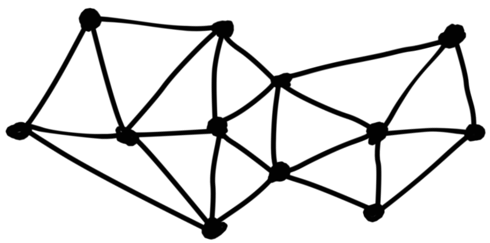
Surfaces as triangulations

- The boundary of surface triangulation is a 1D simplicial complex $M_B = (V_B, E_B)$.



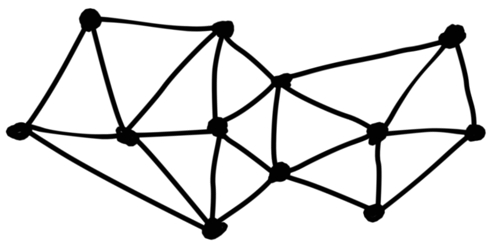
- The interior vertices are denote $V_I = V \setminus V_B$.

- Definition. $M = (V, E, F)$ is connected if it is connected as a graph (V, E). It is k-connected if it cannot be disconnected by removing $k - 1$ vertices.



Surfaces as triangulations

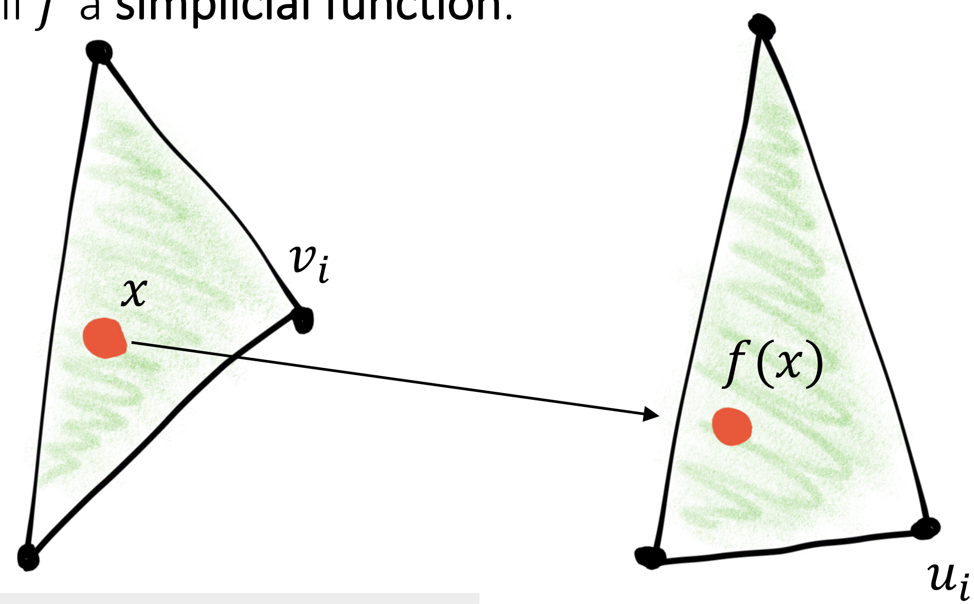
- Lemma [Floater '03]. If $M = (V, E, F)$ is 3-connected then any interior vertex can be connected to any other vertex (including boundary) with an interior path.



Discrete mappings

- Definition. A **simplicial map** $f: M \to \mathbb{R}^d$ is the unique piecewise-linear extension of a vertex map $f_V: V \to \mathbb{R}^d$.

- When $d = 1$ we call f a **simplicial function**.



$$x = \sum_i \lambda_i v_i \quad \mapsto \quad f(x) = \sum_i \lambda_i u_i$$

$$\lambda_i \geq 0, \sum_i \lambda_i = 1$$

The discrete mapping problem

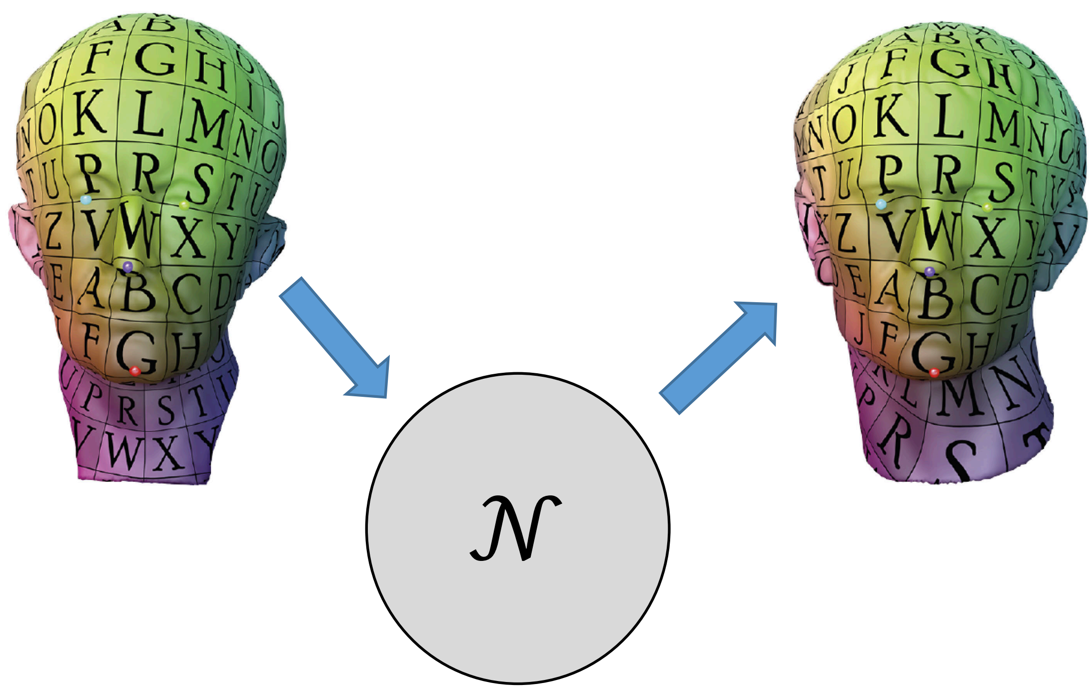
- Problem. Given two topologically equivalent surface triangulations M_1, M_2 and a set of corresponding landmarks $\{(x_i, y_i)\}_{i \in I} \subset M_1 \times M_2$ compute a “nice” simplicial homeomorphism $f\colon M_1 \to M_2$.

The discrete mapping problem

The discrete mapping problem

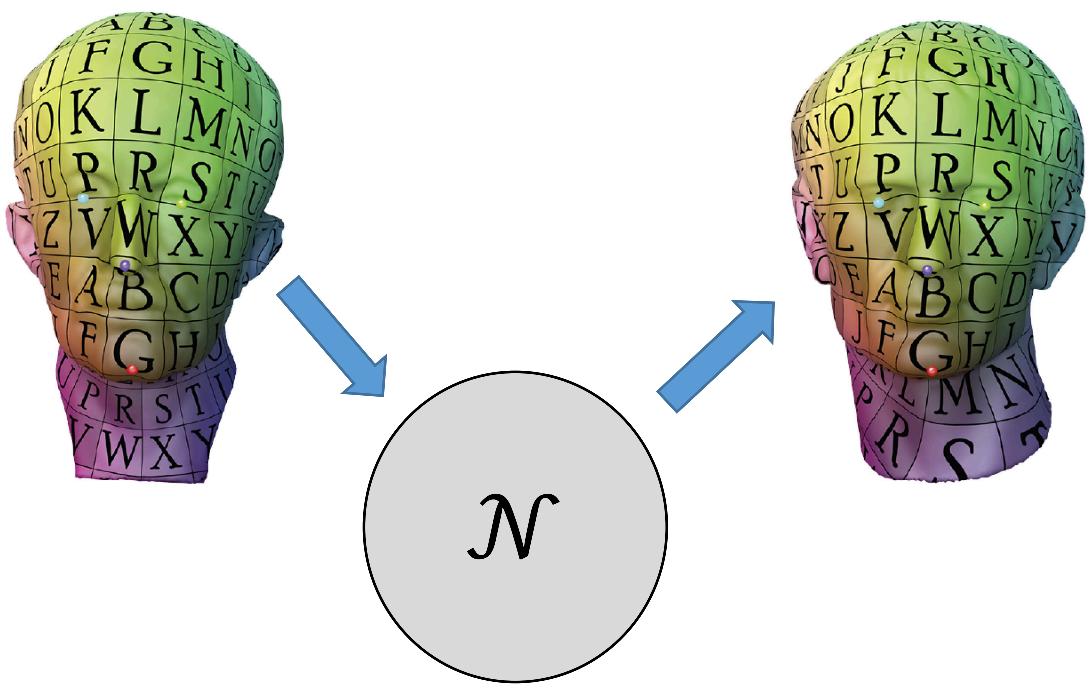
- Difficulty. Requires finding a common isomorphic common triangulation, a combinatorial problem!

- Idea. Consider mapping M_1, M_2 to a canonical domain $\mathcal{N}$, $f_1 : M_1 \to \mathcal{N}$ and $f_2 : M_2 \to \mathcal{N}$ and construct f as $f = f_2^{-1} \circ f_1$.



The discrete mapping problem

- Two questions:
  - How to choose $\mathcal{N}$?
  - How to compute the simplicial map onto $\mathcal{N}$?



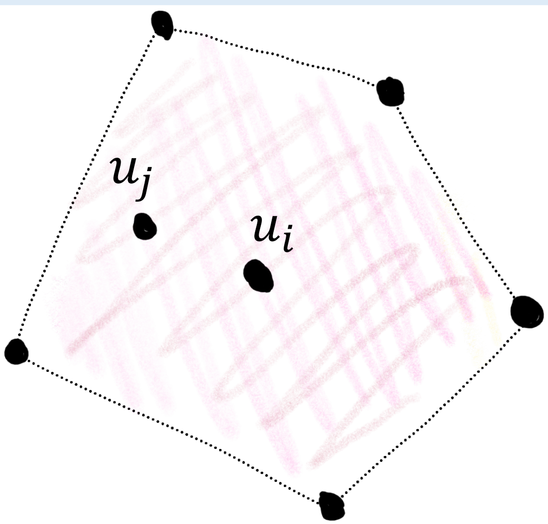
Convex combination mappings

- A technique to map a surface triangulation to $\mathbb{R}^2$.

- Definition. Given a selection of a weight per edge $w_{ij} > 0$, a **convex combination mapping** $f: M \to \mathbb{R}^2$ is a simplicial map mapping each interior vertex $v_i \in V_I$ to a planar point $u_i \in \mathbb{R}^2$ so that

$$\sum_{j \in \mathfrak{N}_i} w_{ij}\left(u_j - u_i\right) = 0,$$

where $\mathfrak{N}_i = \{j | e_{ij} \in E\}$.

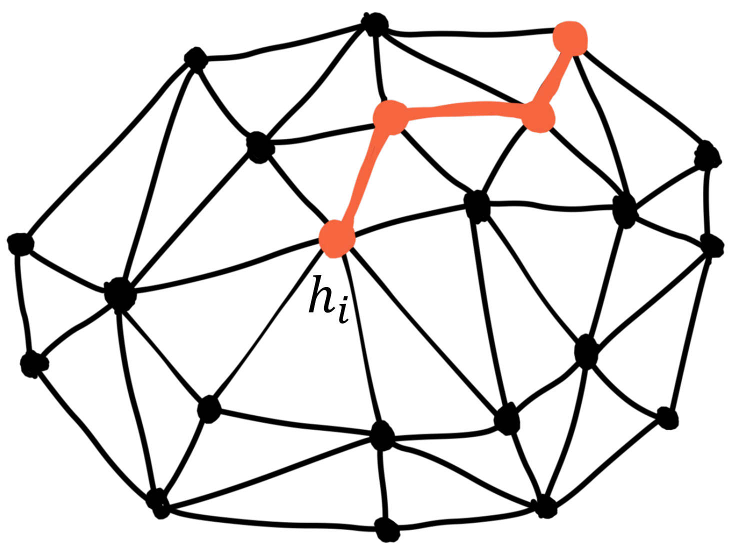


- This property is called **convex combination property** ↔ mean value property.

Discrete maximum principle

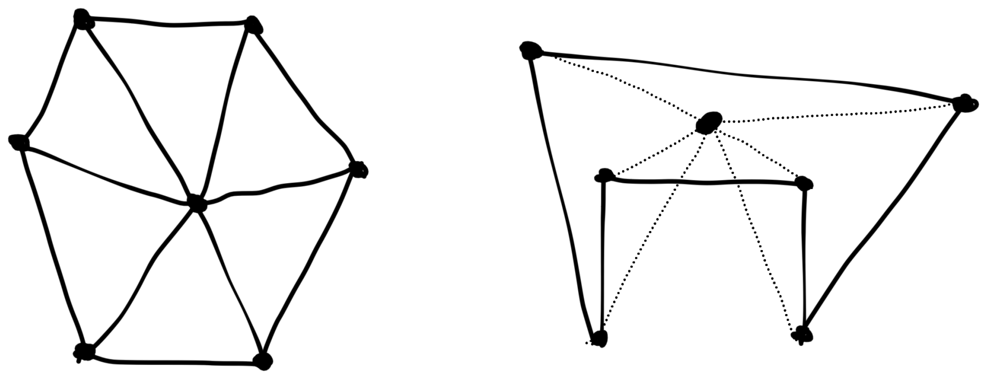
- The convex combination property is a discrete version of the mean value property of hamornic functions.

- Theorem (Discrete maximum principle). Let $h: M \to \mathbb{R}$ be a convex combination function and M a 3-connected surface triangulation. Let $v_i \in V_I$. Then,
  If $h_i = \min_j h_j$ or $h_i = \max_j h_j$ then h is constant.
  In particular h achieves its extreme point on the boundary.



Convex combination mappings

- CCM are in general **not homeomorphisms**, e.g., the constant CCM.

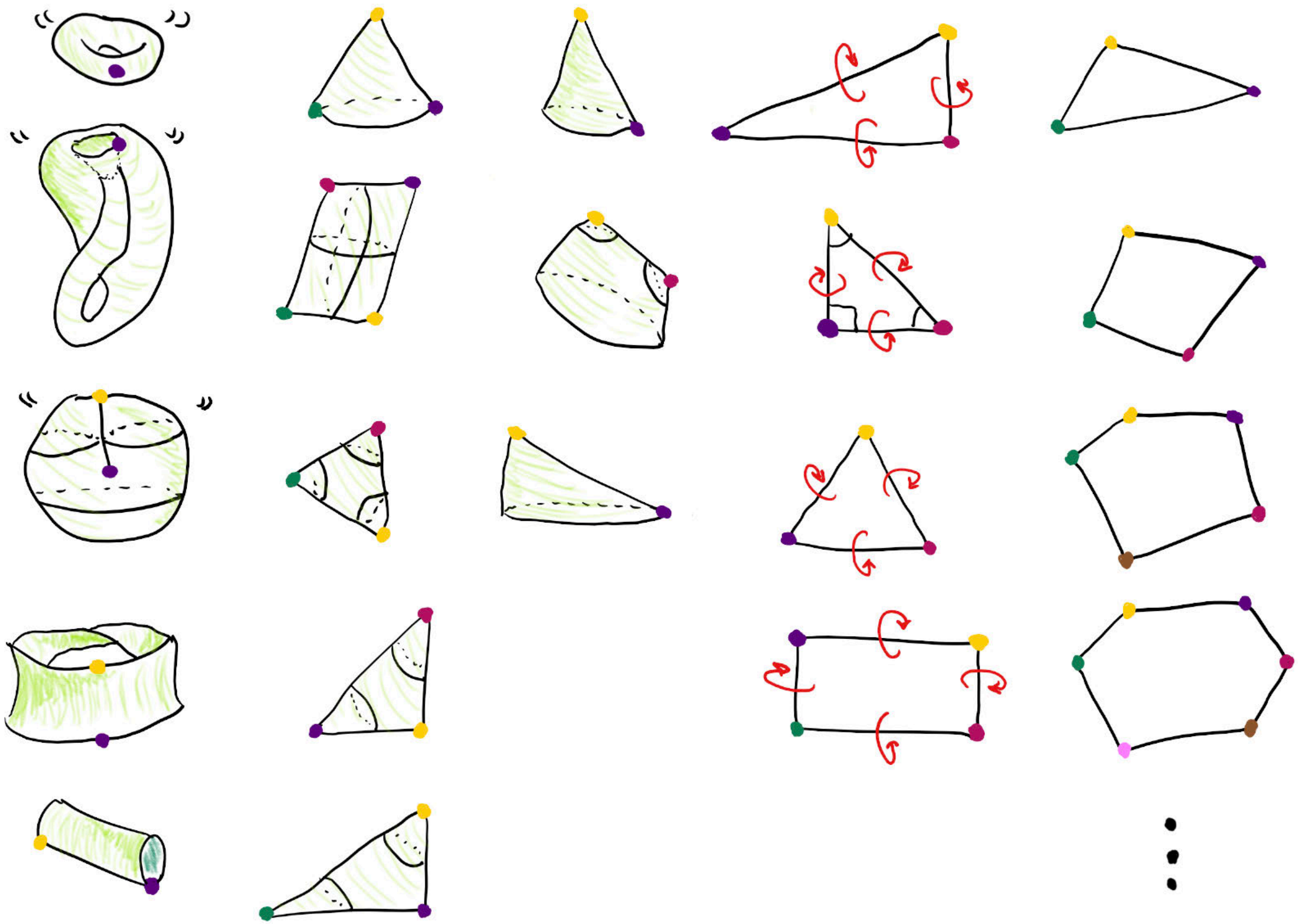


- However, with certain **boundary conditions and target domains $\mathcal{N}$** CCM are guaranteed to be homeomorphic.

- We will explore a family of such target domains:

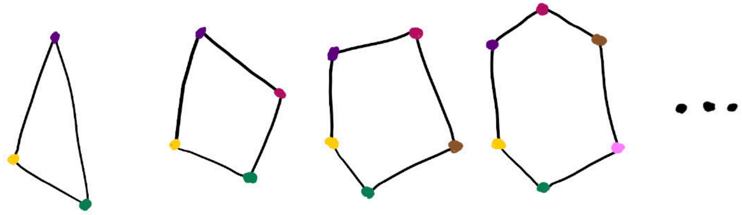
$$\mathcal{F} = \{\mathcal{N}\}$$

$\mathcal{F}$ for homeomorphic CCM



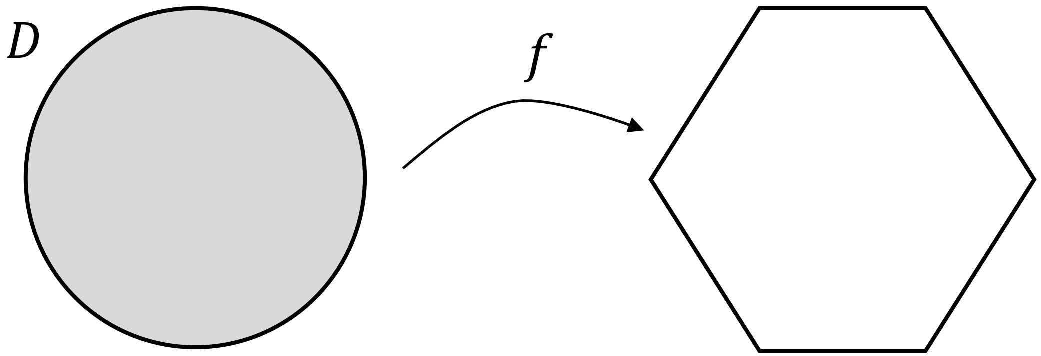
First members of $\mathcal{F}$

- $\mathcal{N}$ is a convex polygonal domain in $\mathbb{R}^2$.



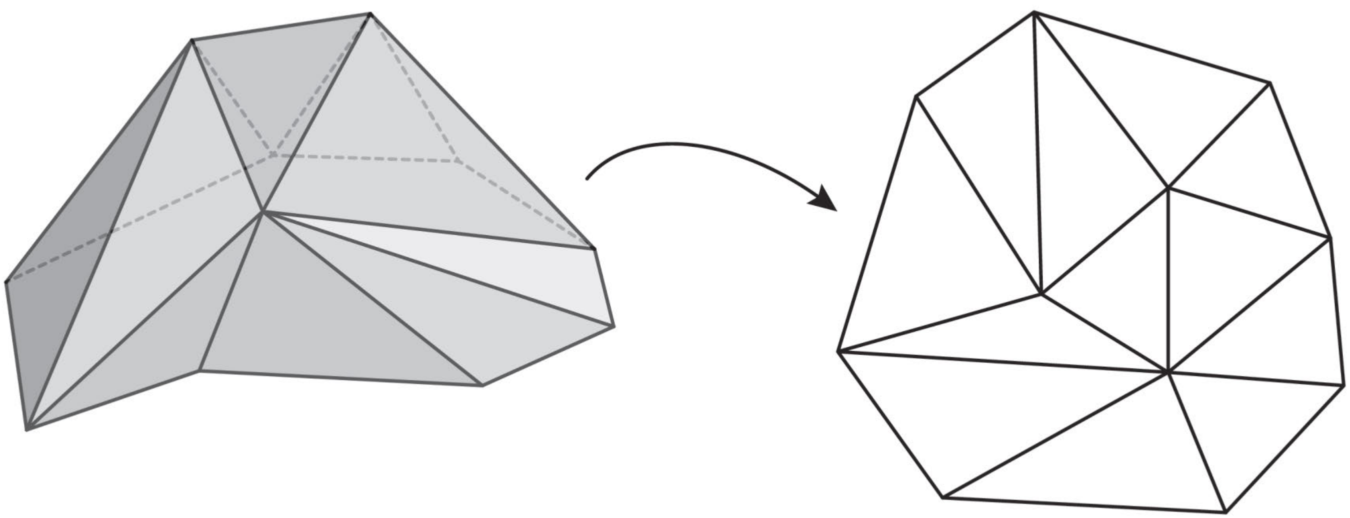
- Hint from analysis:

Theorem [Rado-Kneser-Choquet]: Let $f: D \to \mathbb{R}^2$ be a hamornic map where $f|_{\partial D}$ is a homeomorphism onto the boundary of a convex region. Then, f is homeomorphism.

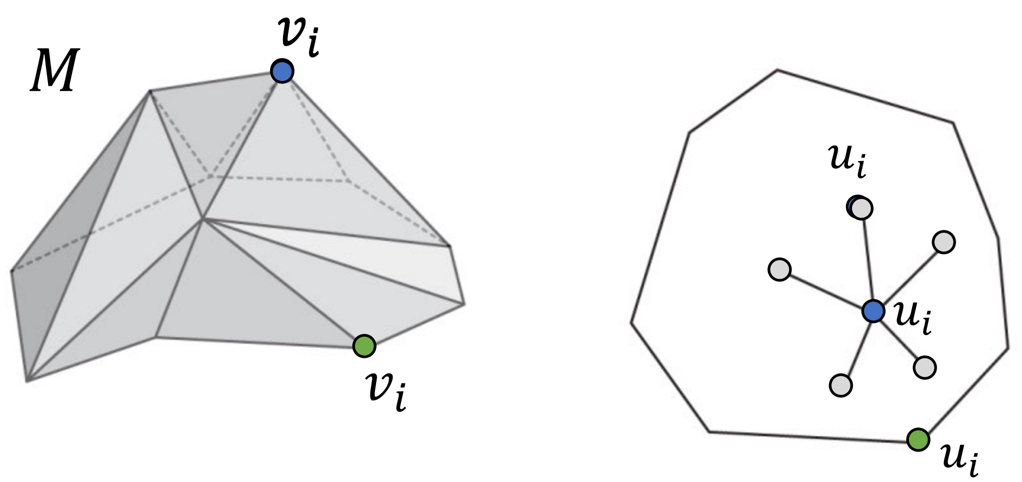


CCM into convex polygonal domain

- Theorem (Tutte, Floater). Let $M = (V, E, F)$ be a 3-connected surface triangulation homeomorphic to a disk. Let $f: M \to \mathbb{R}^2$ be a CCM such that $f|_{M_B}$ is a homeomorphism to a convex polygon enclosing a domain Ω.
  Then, $f: M \to \Omega$ is a homeomorphism.



Computing CCM



$$\sum_{j\in\mathfrak{N}_i} w_{ij}(u_j - u_i) = 0, \quad v_i \in V_I$$
$$u_i = p_i, \quad v_i \in V_B$$

Uniqueness

- Proposition. There is a unique solution to the linear system.

- Proof. Consider a solution to the homogeneous system:

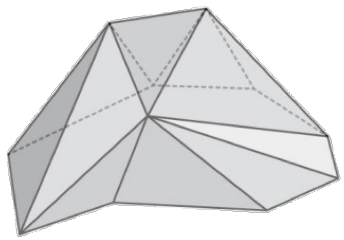
$$\begin{aligned} \textstyle\sum_{j \in \mathfrak{N}_i} w_{ij}\left(u_j - u_i\right) = 0, &\quad v_i \in V_I \\ u_i = 0, &\quad v_i \in V_B \end{aligned}$$

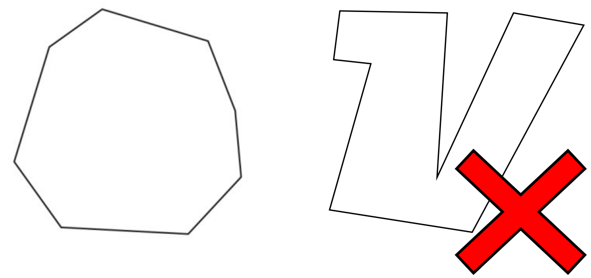
Consider first coordinate x_i of $u_i = (x_i, y_i)$.
This is a CCF hence satisfies discrete maximum principle.
If $x_i \neq 0$ there is a non-zero value at the boundary, contradiction.

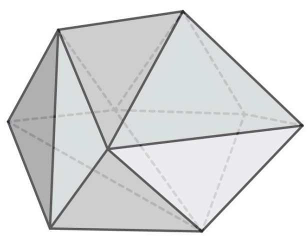
Other members of $\mathcal{F} = \{\mathcal{N}\}$?

Topology

Target domain







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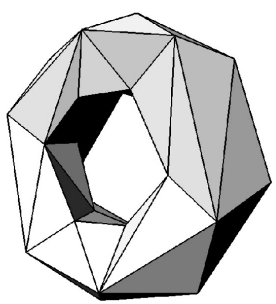

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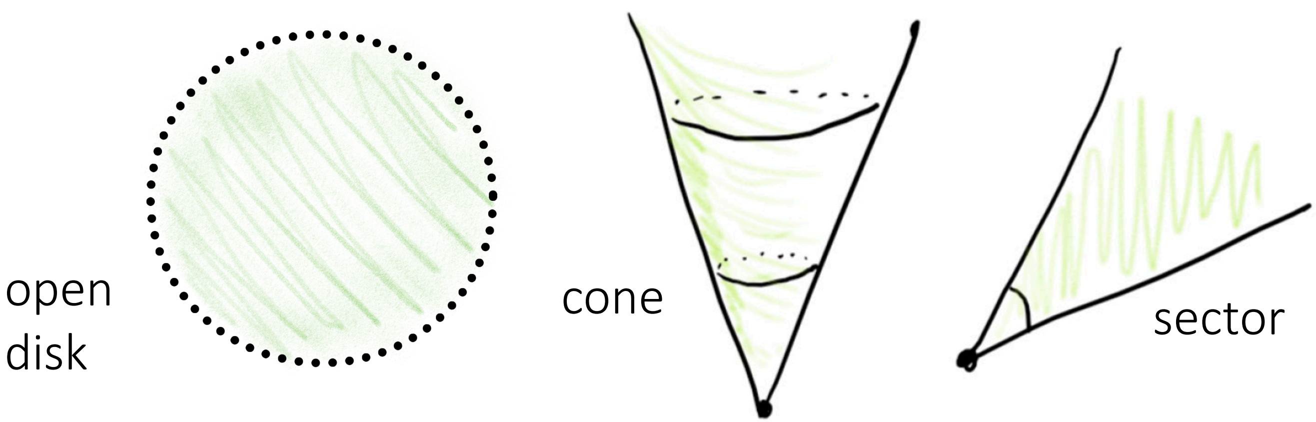
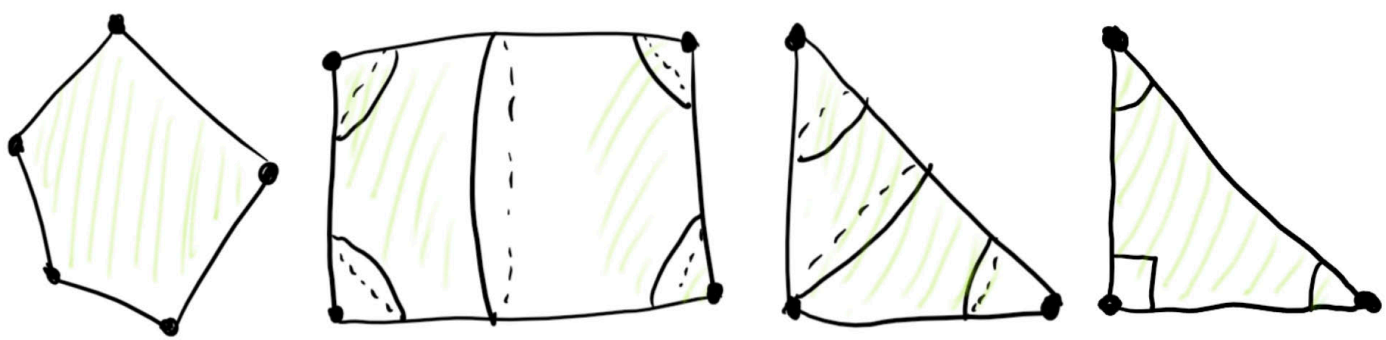
Euclidean cone surfaces

- Definition. a compact surface $\mathcal{N}$ is a **euclidean cone surface** if it is a metric space locally isometric to an open disk, a cone, or a sector and the number of cone points is finite.



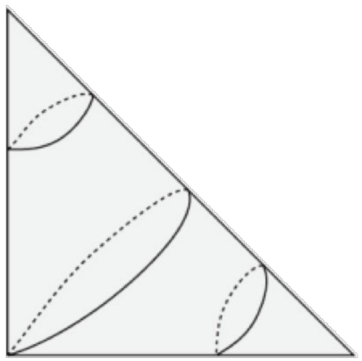


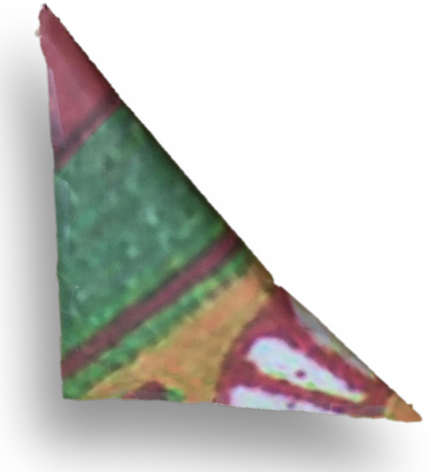
Euclidean orbifolds

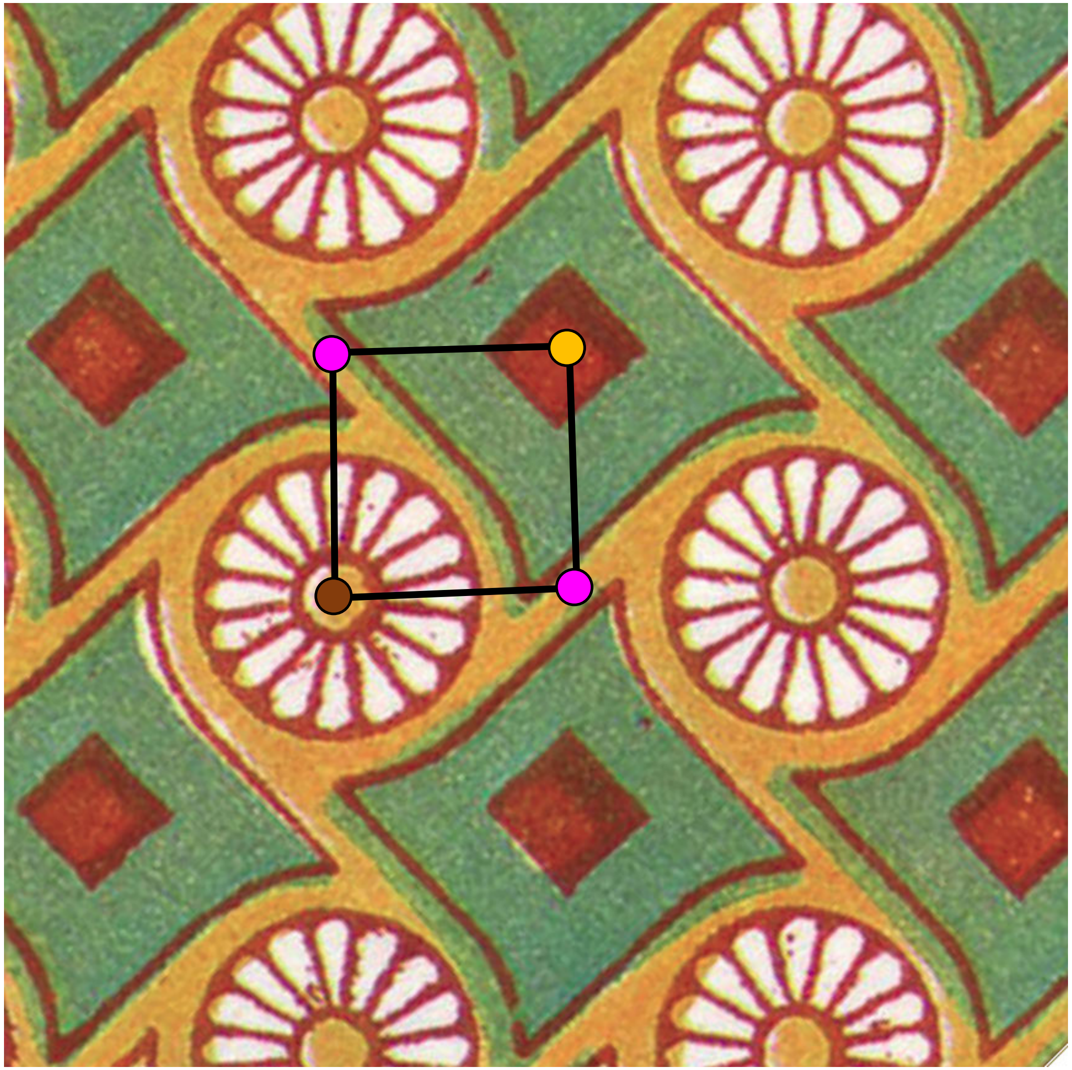
- A subfamily of euclidean cone surfaces.

- Definition. A euclidean orbifold $\mathcal{N}$ is a surface defined as the quotient of $\mathbb{R}^2$ by a symmetry wallpaper group G, that is
$$\mathcal{N} = \mathbb{R}^2/G.$$

- The point of $\mathcal{N}$ are the orbits of G, that is $[u] = \{g(u)|g \in G\}$.

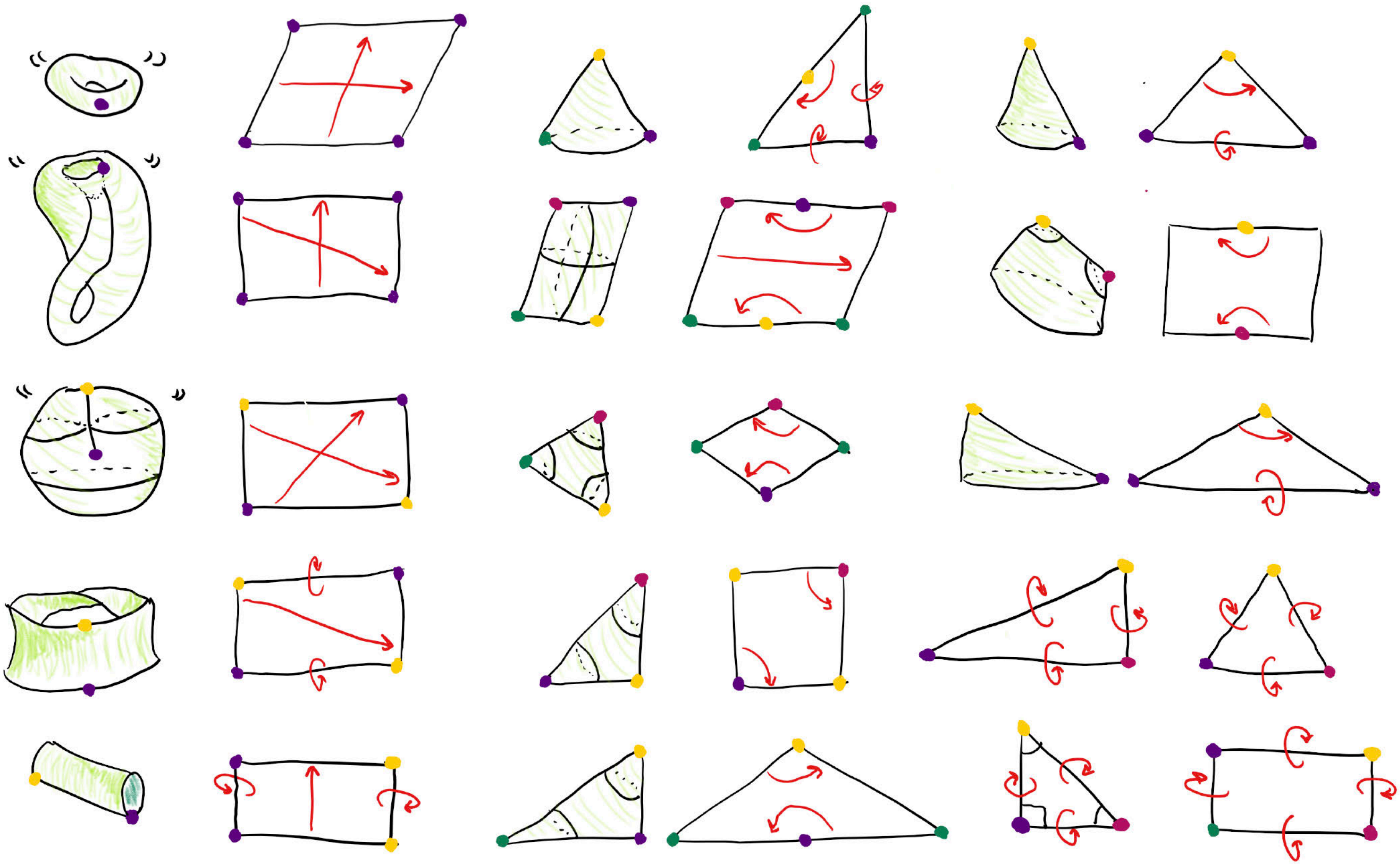






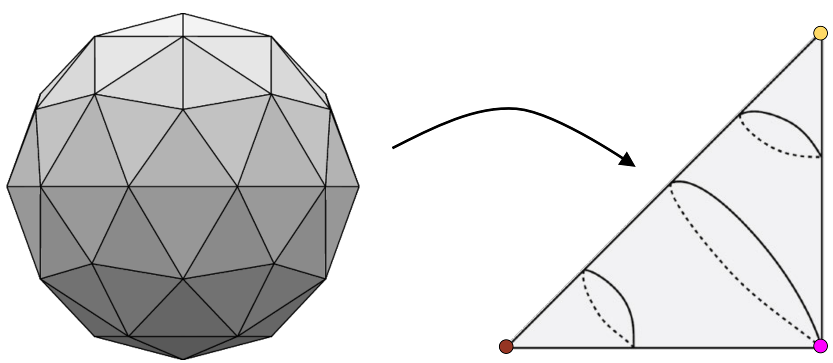
Symmetry of things [Strauss, Burgiel, Conway]

Euclidean orbifolds and their fundamental domains



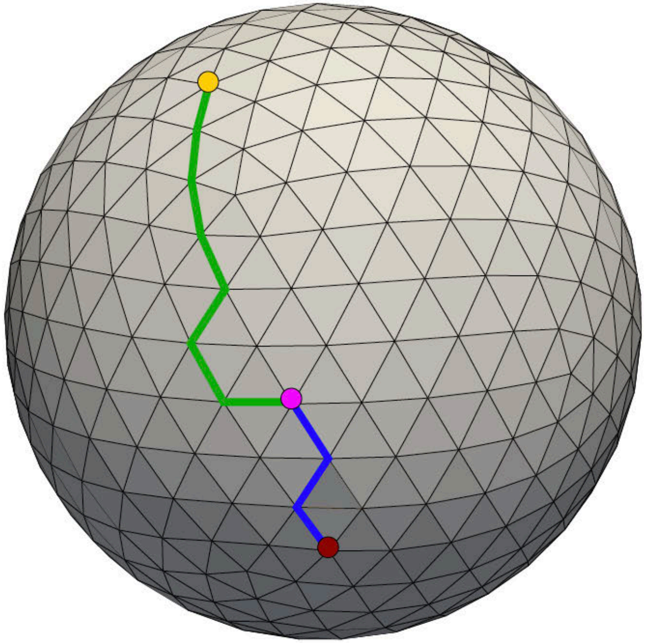
CCM into euclidean orbifolds

- Theorem (orbifold Tutte). Let $M = (V, E, F)$ be a 3-connected surface triangulation homeomorphic to one of the euclidean orbifolds $\mathcal{N}$ with m cones. Let $C = \{v_c\} \subset V$ be a set of m distinct vertices.
  Let $f: M \to \mathcal{N}$ be a CCM such that the $f|_C$ is a bijection between C and the cones of $\mathcal{N}$. Then, $f: M \to \Omega$ is a homeomorphism.



Computing CMM into an orbifold

- First, cut $M = (V, E, F)$ to a disk-type triangulation $M' = (V', E', F')$.
- Second, compute a simplicial map $s: M' \to \mathbb{R}^2$ as follows.



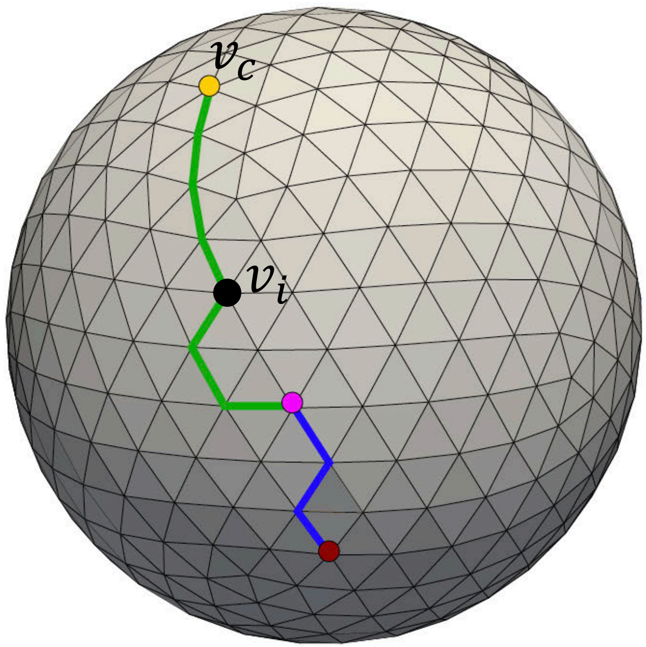
Computing CMM into an orbifold

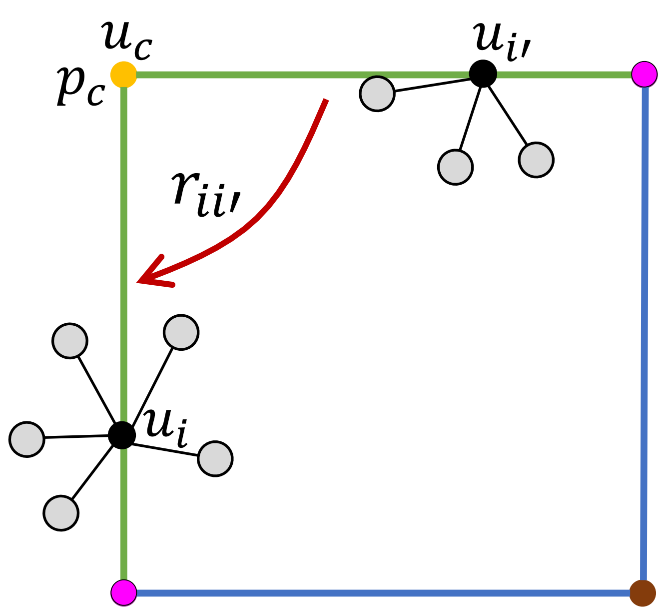
$$\sum_{j\in N_i} w_{ij}(u_j - u_i) = 0$$

$$u_c = p_c$$

$$(u_i - p_c) = r_{ii'}(u_{i'} - p_c)$$

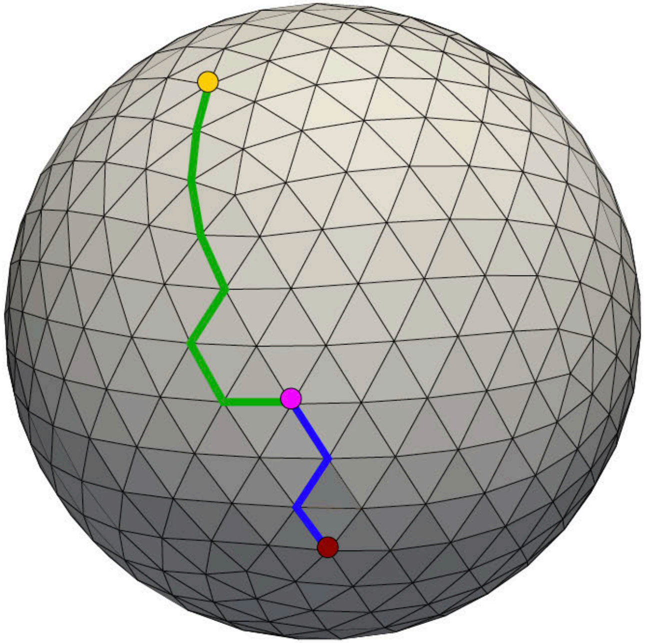
$$\sum_{j\in N_i} w_{ij}(u_j - u_i) + \sum_{j\in N_{i'}} w_{i'j} r_{ii'}(u_j - u_{i'}) = 0$$

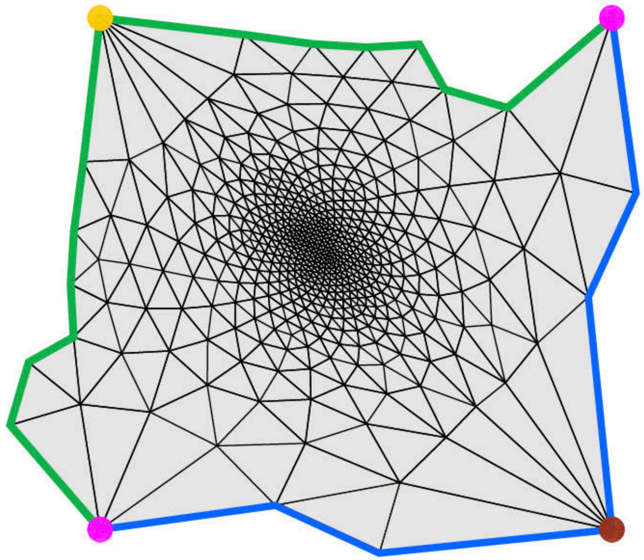


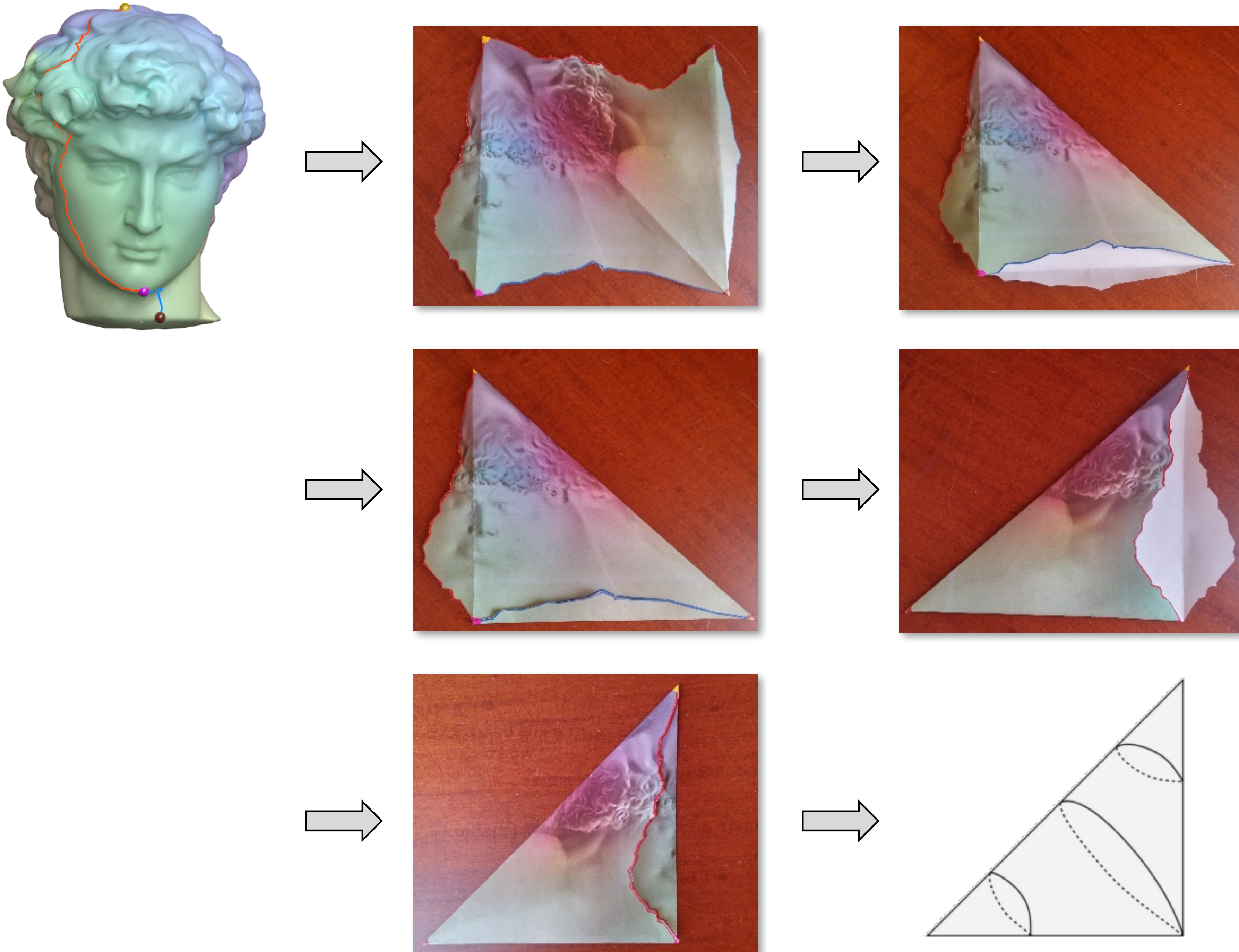


Computing CMM into an orbifold

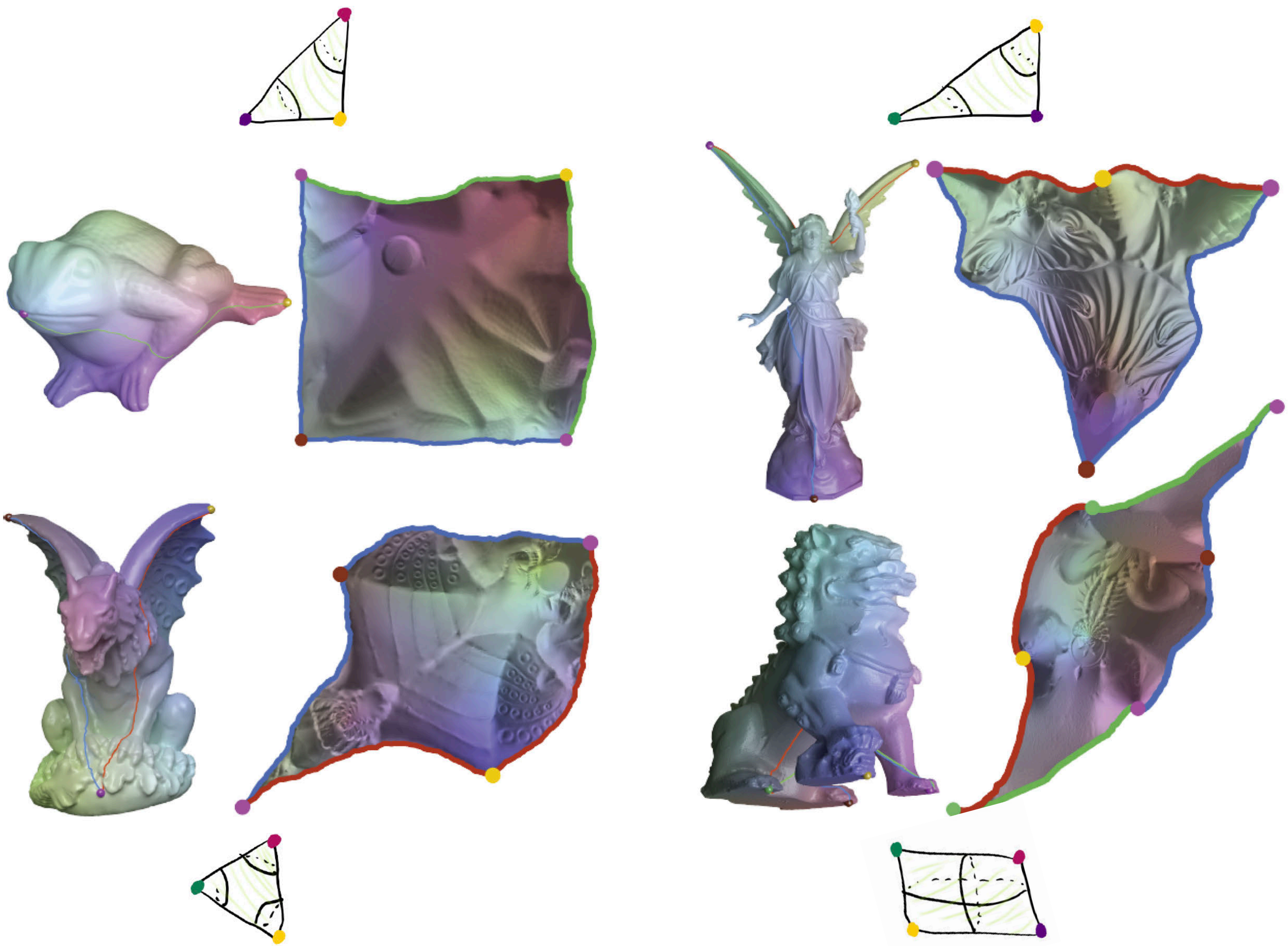
- Lastly, the map $f: M \to \mathcal{N}$ is defined by $f(x) = [s(x)]$.







Example of orbifold CCM



Homeomorphism

- We will outline the idea of the proof.
- Let $s: M' \to \mathbb{R}^2$ be the solution to the linear system previous described.

- Step 1. Build a branched cover M'' to M by stitching copies of M' according to the group G. Consider the extension $s: M'' \to \mathbb{R}^2$.

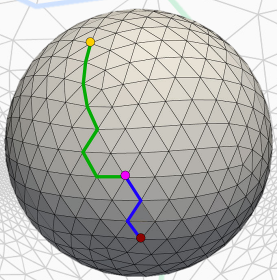
- Step 2. Show $s: M'' \to \mathbb{R}^2$ does not degenerate and maintains the orientation of at-least one triangle.

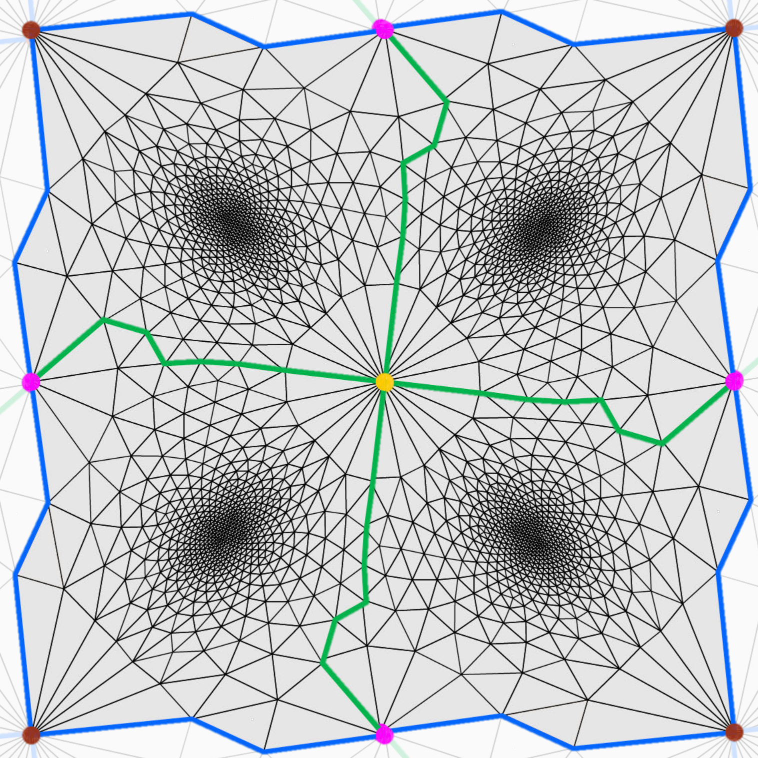
- Step 3. If s does not degenerate and maintains orientation of a triangle, it will also not degenerate nor flip orientation of any neighbor triangle.

- Step 4. If $s: M'' \to \mathbb{R}^2$ maintains orientation of all triangles it is a homeomorphism. Consequently $f: M \to \mathcal{N}$ is a homeomorphism.

Homeomorphism

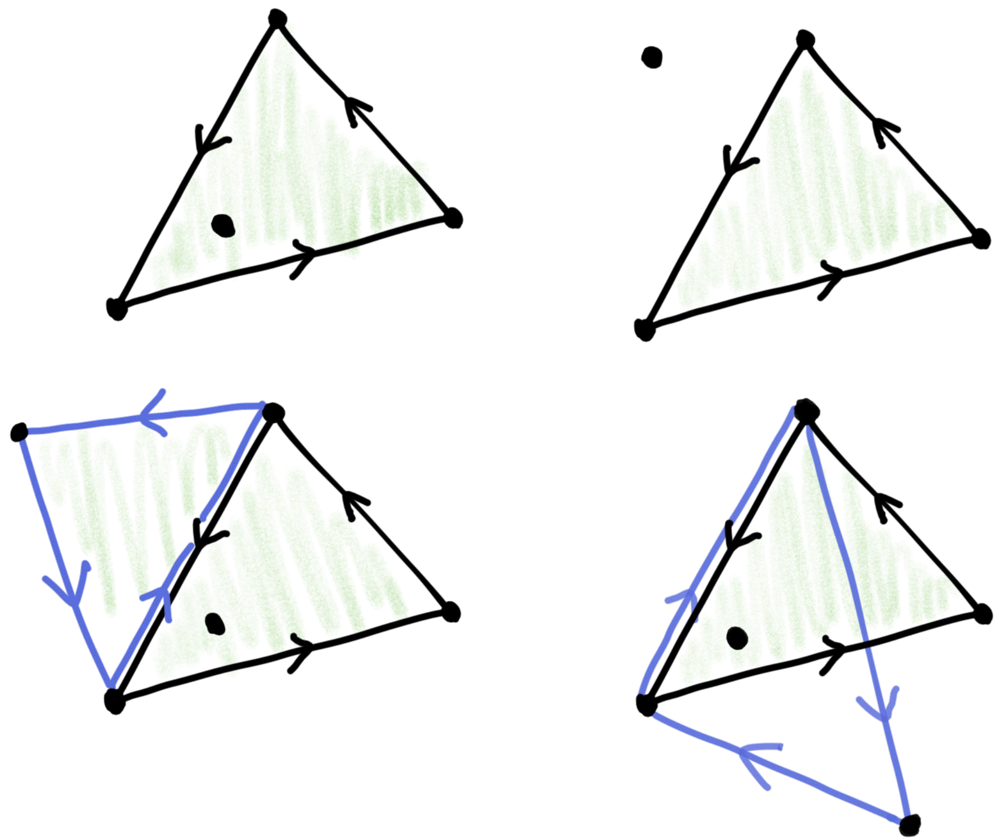
- Step 1. Build a branched cover M'' to M by stitching copies of M' according to the group G. Consider the extension $s: M'' \rightarrow \mathbb{R}^2$. All vertices satisfy the CCP.

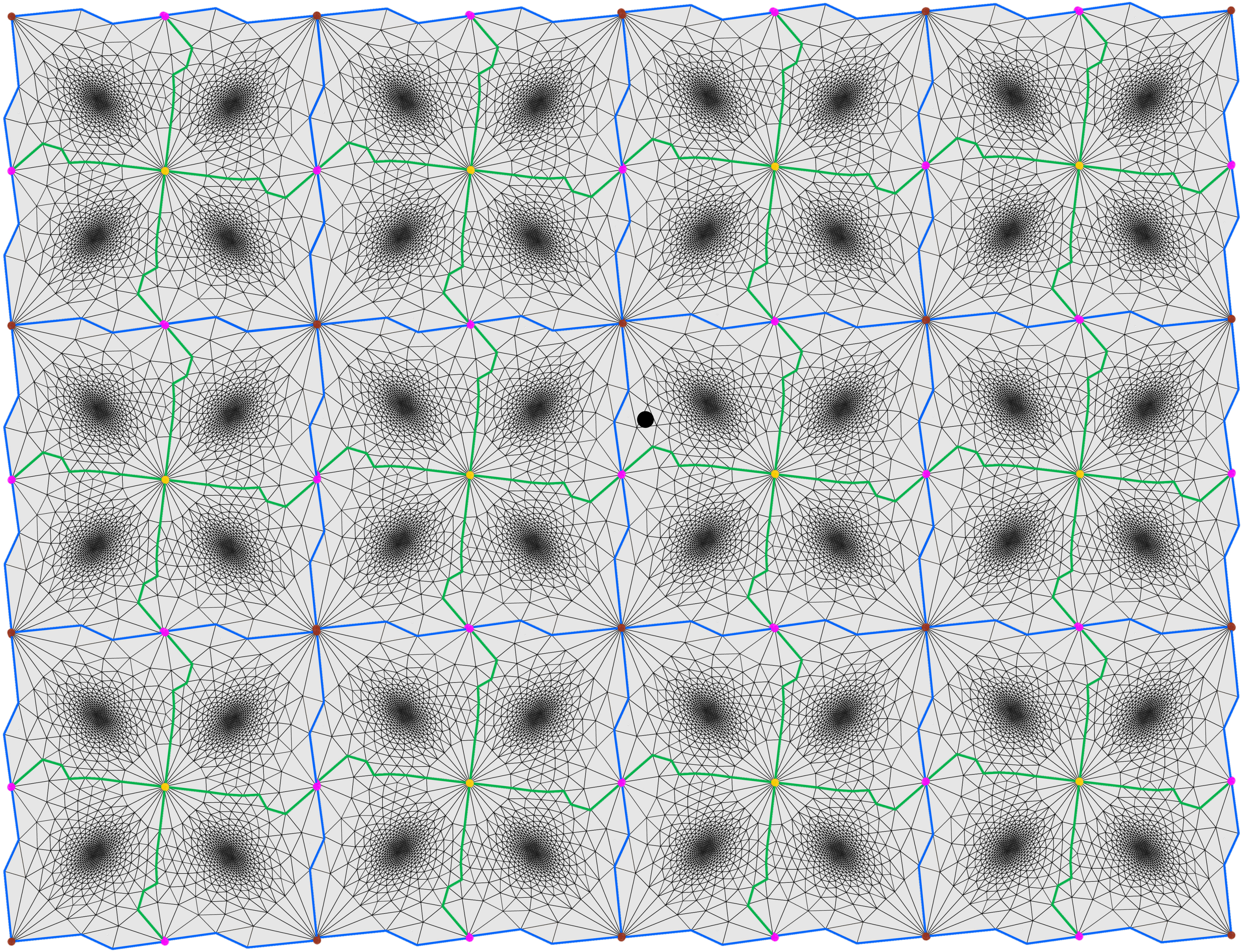




Homeomorphism

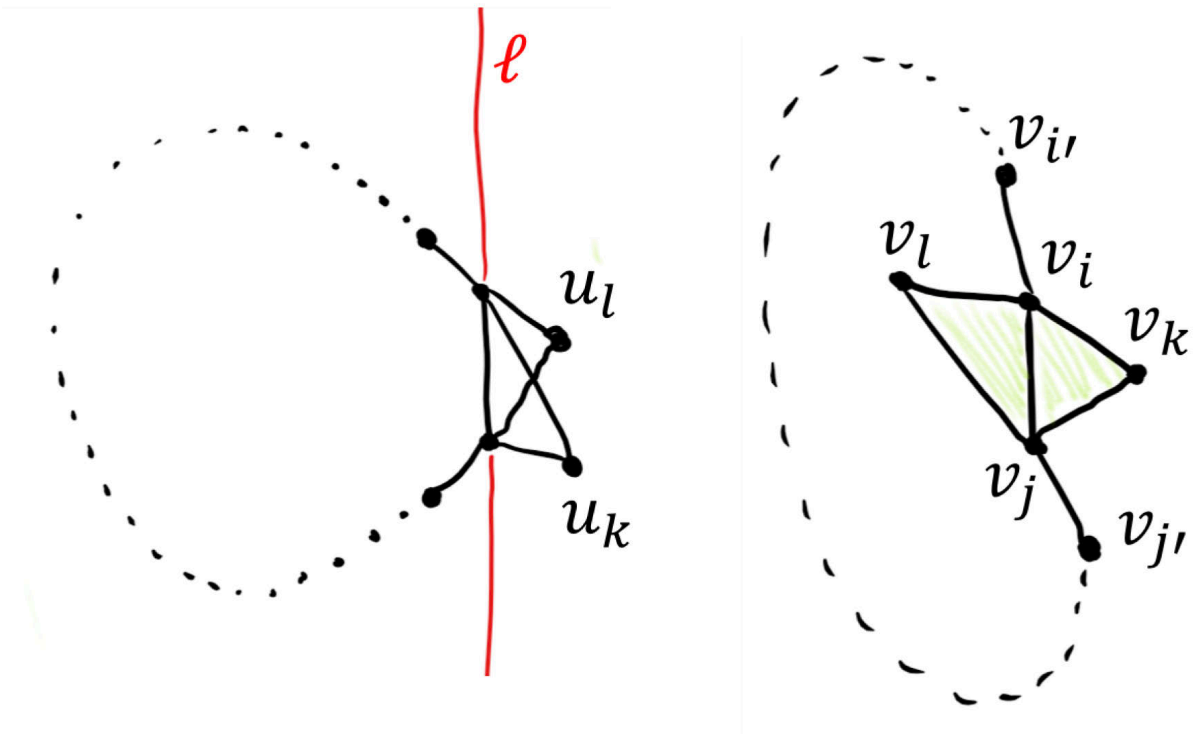
- Step 2. We will show a stronger claim. Every (generic) point in the plane is covered by at-least one positively oriented triangle.





Homeomorphism

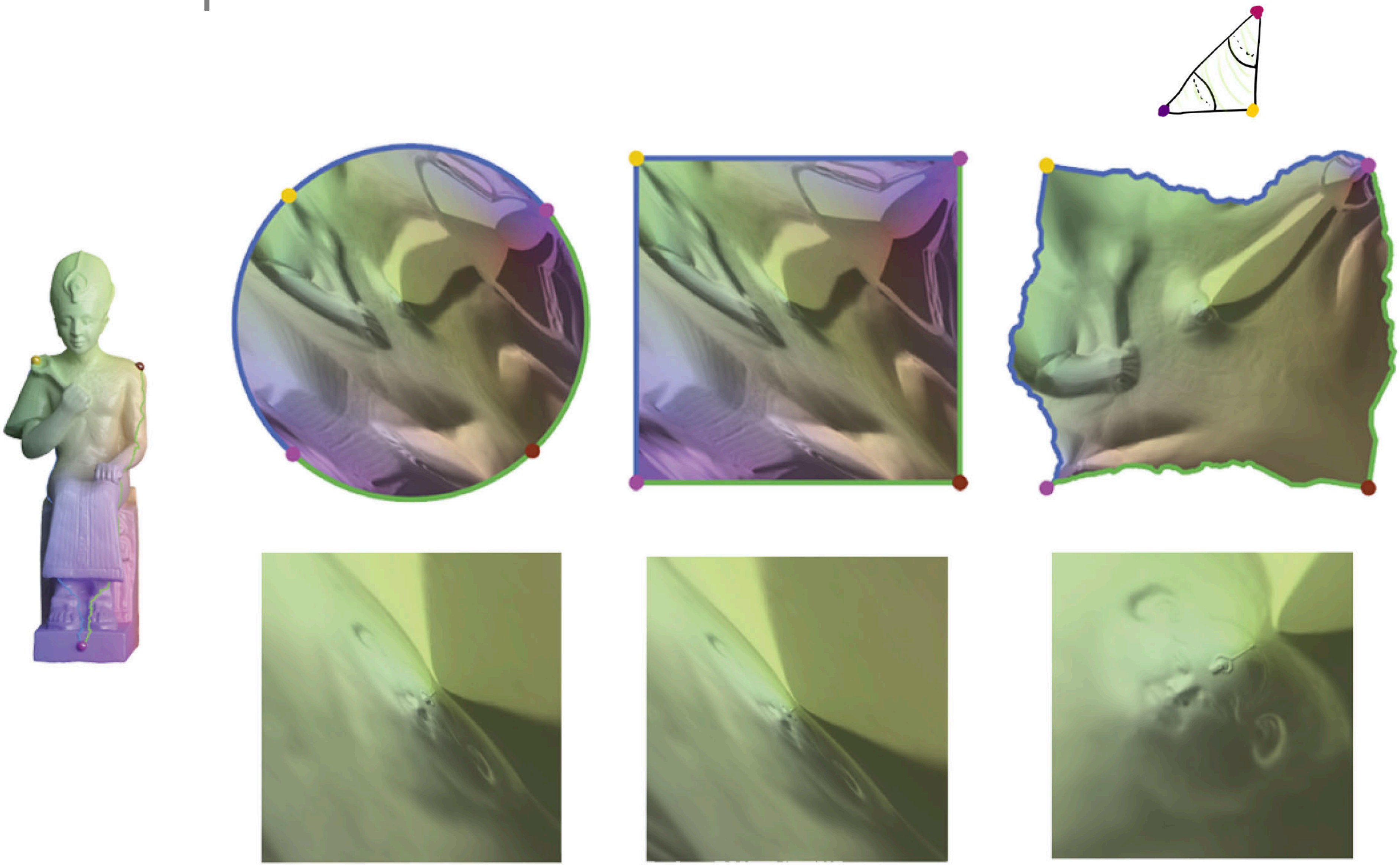
- Step 3. If s does not degenerate and maintains orientation of a triangle, it will also not degenerate nor flip any neighbor triangle.



Homeomorphism

- Step 4. If $s: M'' \to \mathbb{R}^2$ maintains orientation of all triangles it is a homeomorphism. Consequently $f: M \to \mathcal{N}$ is a homeomorphism.
- Repeat winding number argument but now we know that all triangles are positively oriented.

Comparison of CCM



Variational principle

- When $w_{ij} = w_{ji}$ there exists a variational form

$$\begin{aligned} \min \quad & \frac{1}{2}\sum_{e_{ij}} w_{ij}\left(u_j - u_i\right)^2 \\ \text{s.t.} \quad & \text{boundary conditions} \end{aligned}$$

- This energy is called discrete Dirichlet energy, $E_D(u)$.
- A popular choice of weights comes from asking that $E_D(u) = \int_M |\nabla f|$.
- These weights are called cotan weights and $w_{ij} = \cot\alpha_{ij} + \cot\beta_{ij}$.
- The mesh is Delaunay ($\alpha_{ij} + \beta_{ij} < \pi$) iff $w_{ij} > 0$.

Conformality

- The Dirichlet energy satisfies:

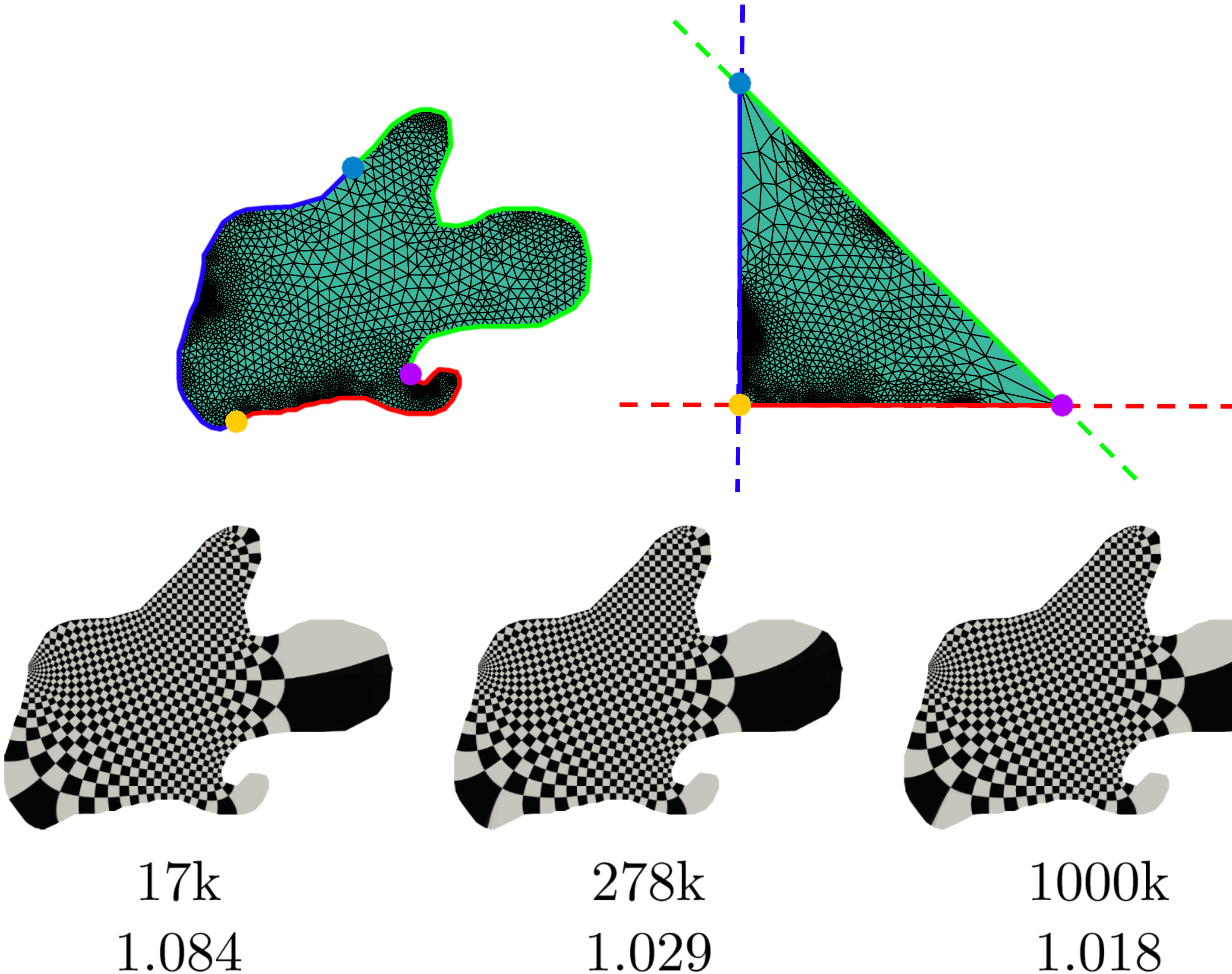
$$E_D(u) = E_C(u) + E_A(u)$$

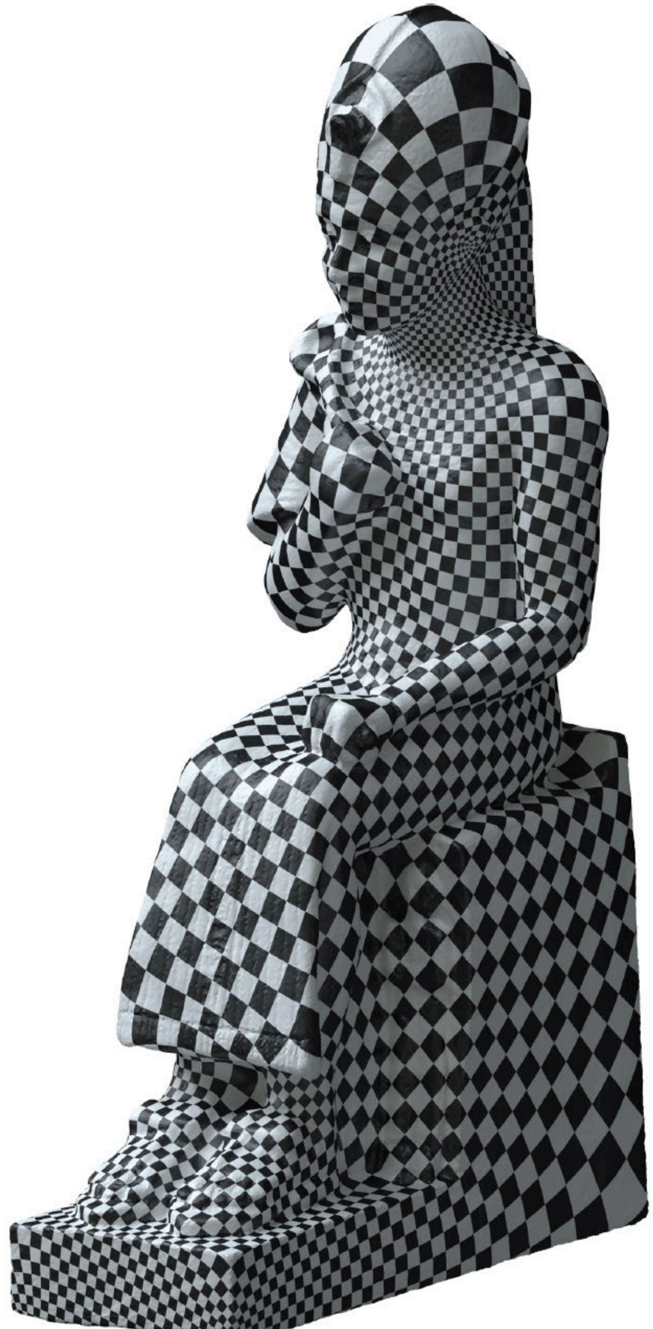
where E_A is the area functional summing positive areas of triangles.

- The orbifold Tutte theorem implies that $E_A(u) = area(\mathcal{N})$ constant.
- Since the number of point constraints matches the degrees of freedom in conformal map we can ask:

Does $f: M \rightarrow \mathcal{N}$ converge to a conformal map under refinement of M?

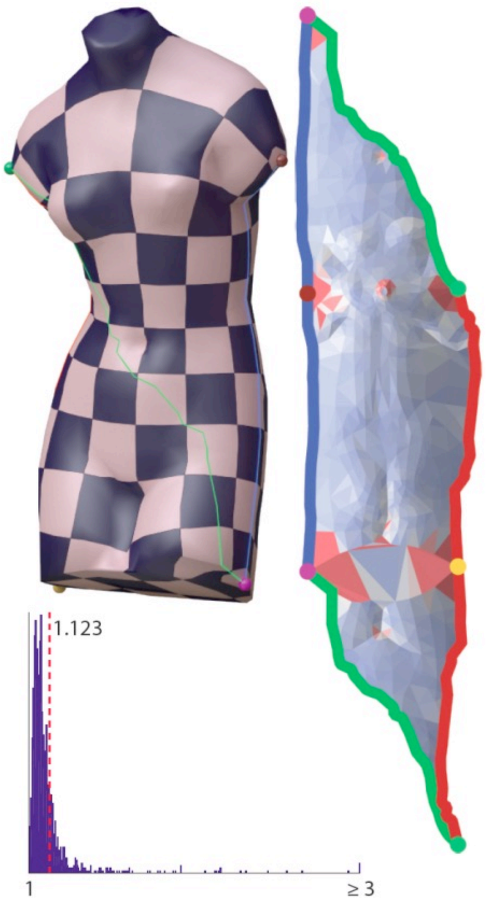
- Theorem. Convergence in H^1 holds for $\mathcal{N}$ a triangle orbifold. If M is Delaunay uniform convergence hold.


17k
1.084
278k
1.029
1000k
1.018

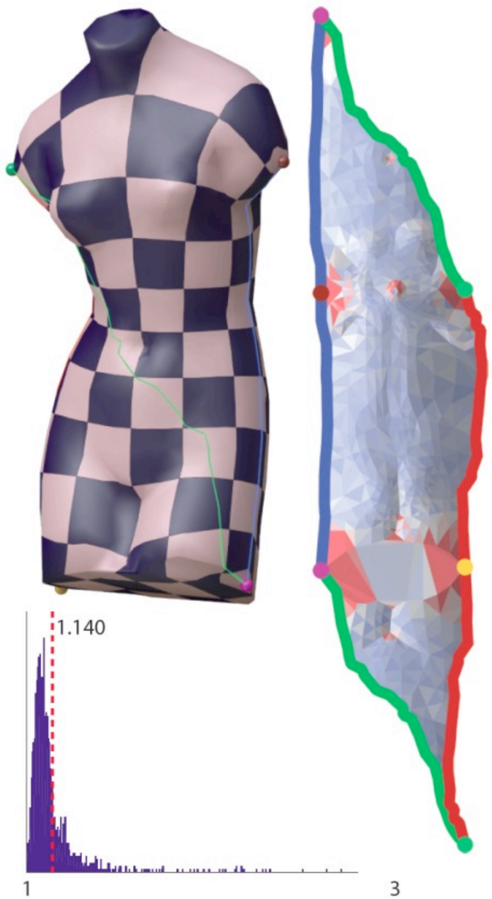


Discrete uniformization

[Springborn et al. 08]



Orbifold-Tutte



Discrete mapping of surfaces

- Back to the discrete mapping problem: we got a solution for up-to 4 landmark constraints.

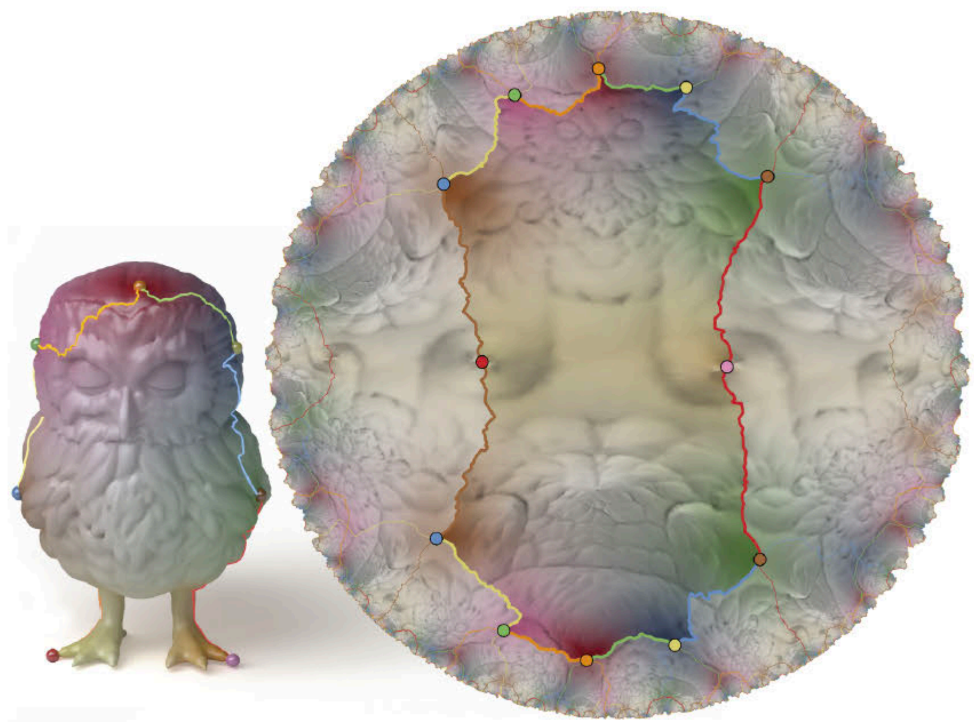
- Discrete extremal quasiconformal maps….?

Open problems

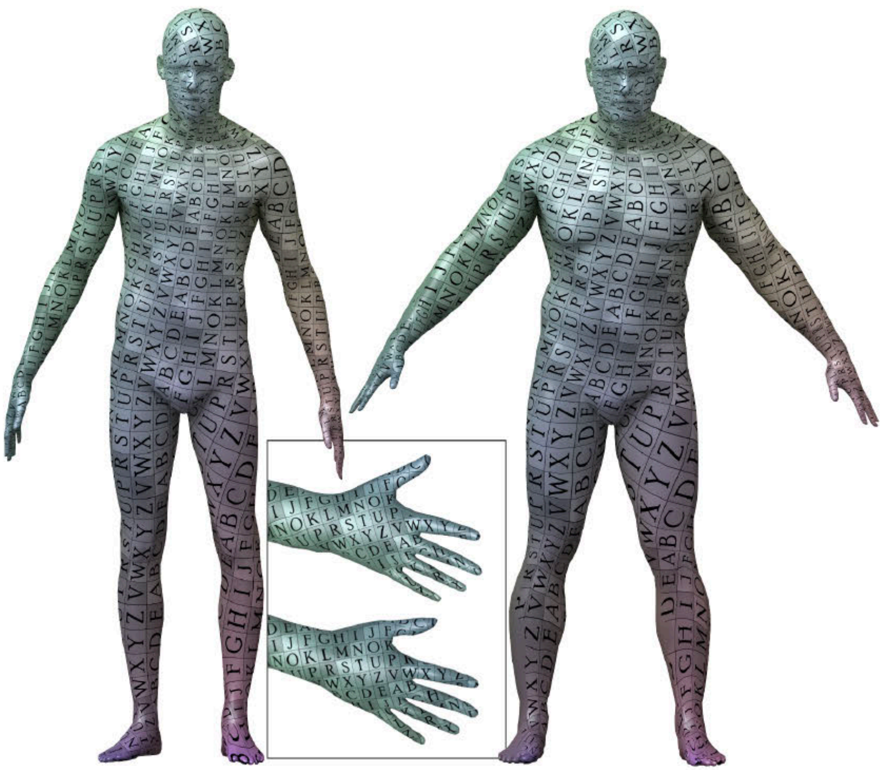
- Problem. Can $\mathcal{F}$ be enlarged?
- I am not aware of such result.
- Problem. Can $\mathcal{F}$ be enlarged under extra conditions?
- Several interesting such results. See notes.

Beyond euclidean

- CCM can be generalized to hyperbolic plane.
- Basic results (Tutte, Orbifold Tutte) still holds.
- Allows infinite number of cones.
- Drawback: no longer a linear model

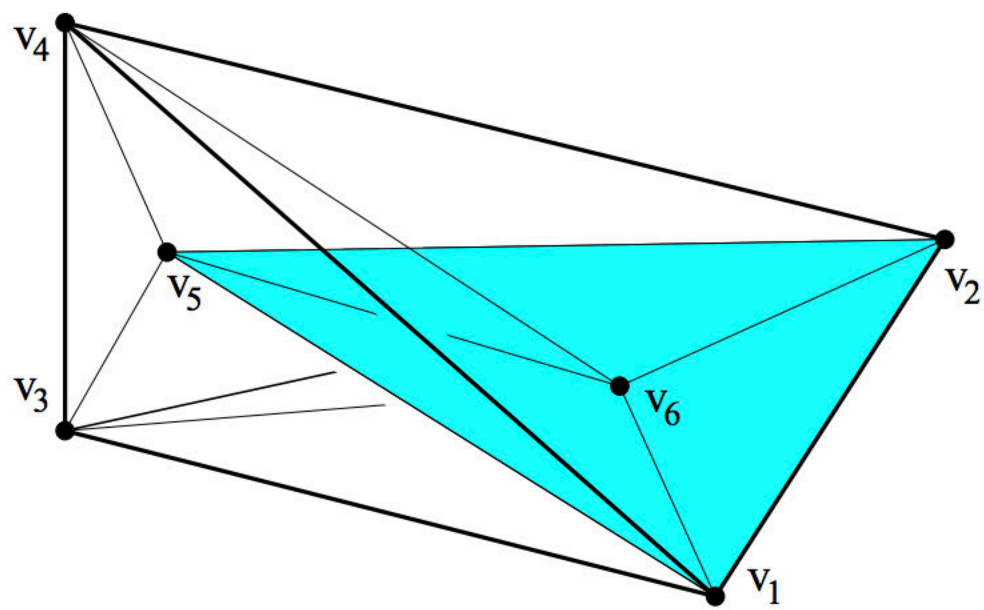


Beyond euclidean



Higher dimensions?

- Counter example to Tutte exists. The following example by [Floater, Pham-Trong].



The end

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