Discrete Mappings

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Surfaces as triangulations

- Triangles stitched to build a surface.
Surfaces as triangulations

- Triangles stitched to build a surface.
- \( M = (V, E, F); \)
- \( V = \{v_i\}, E = \{e_{ij}\}, F = \{f_{ijk}\}. \)
Surfaces as triangulations

- $M = (V, E, F)$

Rules for stitching triangles:
1. $M$ is a simplicial complex.
2. $\text{link}(v_i) = \bigcup \{ f_{ijk} \in F \} e_{jk}$ is a simple closed polygon.
Surfaces as triangulations

- **Definition.** A surface triangulation is a triplet $M = (V, E, F)$ satisfying the stitching rules.

- For surface triangulation with boundary, replace stitching rule 2 with
  
  2. $\text{link}(v_i)$ is a simple (closed or not) polygon.
Surfaces as triangulations

• The boundary of surface triangulation is a 1D simplicial complex $M_B = (V_B, E_B)$.

• The interior vertices are denote $V_I = V \setminus V_B$.

• Definition. $M = (V, E, F)$ is connected if it is connected as a graph $(V, E)$. It is $k$-connected if it cannot be disconnected by removing $k - 1$ vertices.
Surfaces as triangulations

- Lemma [Floater ’03]. If $M = (V, E, F)$ is 3-connected then any interior vertex can be connected to any other vertex (including boundary) with an interior path.
Discrete mappings

• **Definition.** A simplicial map \( f : M \to \mathbb{R}^d \) is the unique piecewise-linear extension of a vertex map \( f_V : V \to \mathbb{R}^d \).

• When \( d = 1 \) we call \( f \) a **simplicial function**.

\[
x = \sum_{i} \lambda_i v_i \quad \Rightarrow \quad f(x) = \sum_{i} \lambda_i u_i
\]

\( \lambda_i \geq 0, \sum_{i} \lambda_i = 1 \)
The discrete mapping problem

• **Problem.** Given two topologically equivalent surface triangulations $M_1, M_2$ and a set of corresponding landmarks $\{(x_i, y_i)\}_{i \in I} \subset M_1 \times M_2$ compute a “nice” simplicial homeomorphism $f : M_1 \to M_2$. 
The discrete mapping problem
The discrete mapping problem

- **Difficulty.** Requires finding a common isomorphic common triangulation, a combinatorial problem!

- **Idea.** Consider mapping $M_1, M_2$ to a canonical domain $\mathcal{N}$, $f_1: M_1 \to \mathcal{N}$ and $f_2: M_2 \to \mathcal{N}$ and construct $f$ as $f = f_2^{-1} \circ f_1$. 
The discrete mapping problem

• Two questions:
  • How to choose $\mathcal{N}$?
  • How to compute the simplicial map onto $\mathcal{N}$?
Convex combination mappings

• A technique to map a surface triangulation to $\mathbb{R}^2$.

• **Definition.** Given a selection of a weight per edge $w_{ij} > 0$, a convex combination mapping $f : M \rightarrow \mathbb{R}^2$ is a simplicial map mapping each interior vertex $v_i \in V_i$ to a planar point $u_i \in \mathbb{R}^2$ so that

$$\sum_{j \in \mathcal{N}_i} w_{ij}(u_j - u_i) = 0,$$

where $\mathcal{N}_i = \{j | e_{ij} \in E\}$.

• This property is called **convex combination property** $\iff$ **mean value property**.
Discrete maximum principle

• The convex combination property is a discrete version of the mean value property of harmonic functions.

• Theorem (Discrete maximum principle). Let $h : M \to \mathbb{R}$ be a convex combination function and $M$ a 3-connected surface triangulation. Let $v_i \in V_i$. Then, if $h_i = \min_j h_j$ or $h_i = \max_j h_j$ then $h$ is constant.

In particular $h$ achieves its extreme point on the boundary.
Convex combination mappings

- CCM are in general **not homeomorphisms**, e.g., the constant CCM.

- However, with certain **boundary conditions and target domains** $\mathcal{N}$ CCM are guaranteed to be homeomorphic.

- We will explore a family of such target domains:

$$\mathcal{F} = \{ \mathcal{N} \}$$
\( \mathcal{F} \) for homeomorphic CCM
First members of $\mathcal{F}$

- $\mathcal{N}$ is a convex polygonal domain in $\mathbb{R}^2$.

- Hint from analysis:

**Theorem [Rado-Kneser-Choquet]:** Let $f: D \to \mathbb{R}^2$ be a harmonic map where $f|_{\partial D}$ is a homeomorphism onto the boundary of a convex region. Then, $f$ is homeomorphism.
Theorem (Tutte, Floater). Let $M = (V, E, F)$ be a 3-connected surface triangulation homeomorphic to a disk. Let $f : M \to \mathbb{R}^2$ be a CCM such that $f|_{M_B}$ is a homeomorphism to a convex polygon enclosing a domain $\Omega$. Then, $f : M \to \Omega$ is a homeomorphism.
Computing CCM

\[ \sum_{j \in \mathcal{N}_i} w_{ij} (u_j - u_i) = 0, \quad v_i \in V_I \]
\[ u_i = p_i, \quad v_i \in V_B \]
Uniqueness

- **Proposition.** There is a unique solution to the linear system.

- **Proof.** Consider a solution to the homogeneous system:

\[
\sum_{j \in \mathbb{N}_i} w_{ij} (u_j - u_i) = 0, \quad \nu_i \in V_I \\
u_i = 0, \quad \nu_i \in V_B
\]

Consider first coordinate \(x_i\) of \(u_i = (x_i, y_i)\).
This is a CCF hence satisfies discrete maximum principle.
If \(x_i \neq 0\) there is a non-zero value at the boundary, contradiction.
Other members of $\mathcal{F} = \{\mathcal{N}\}$?
Euclidean cone surfaces

- **Definition.** A compact surface $\mathcal{N}$ is a **Euclidean cone surface** if it is a metric space locally isometric to an open disk, a cone, or a sector and the number of cone points is finite.
Euclidean orbifolds

• A subfamily of euclidean cone surfaces.

• **Definition.** A euclidean orbifold $\mathcal{N}$ is a surface defined as the quotient of $\mathbb{R}^2$ by a symmetry wallpaper group $G$, that is

$$\mathcal{N} = \mathbb{R}^2 / G.$$  

• The point of $\mathcal{N}$ are the orbits of $G$, that is $[u] = \{g(u) | g \in G\}$. 
Euclidean orbifolds

Symmetry of things [Strauss, Burgiel, Conway]
Euclidean orbifolds and their fundamental domains
CCM into euclidean orbifolds

**Theorem (orbifold Tutte).** Let $M = (V, E, F)$ be a 3-connected surface triangulation homeomorphic to one of the euclidean orbifolds $\mathcal{N}$ with $m$ cones. Let $C = \{v_c\} \subset V$ be a set of $m$ distinct vertices. Let $f: M \to \mathcal{N}$ be a CCM such that the $f|_C$ is a bijection between $C$ and the cones of $\mathcal{N}$. Then, $f: M \to \Omega$ is a homeomorphism.
Computing CMM into an orbifold

- First, cut $M = (V, E, F)$ to a disk-type triangulation $M' = (V', E', F')$.
- Second, compute a simplicial map $s: M' \rightarrow \mathbb{R}^2$ as follows.
Computing CMM into an orbifold

\[ \sum_{j \in N_i} w_{ij}(u_j - u_i) = 0 \]

\[ u_c = p_c \]

\[ (u_i - p_c) = r_{ii'}(u_{i'} - p_c) \]

\[ \sum_{j \in N_i} w_{ij}(u_j - u_i) + \sum_{j \in N_{ii'}} w_{ij} r_{ii'}(u_j - u_{i'}) = 0 \]
Computing CMM into an orbifold

- Lastly, the map $f: M \rightarrow \mathcal{N}$ is defined by $f(x) = [s(x)]$. 
Example of orbifold CCM
We will outline the idea of the proof.
Let $s: M' \to \mathbb{R}^2$ be the solution to the linear system previous described.

Step 1. Build a branched cover $M''$ to $M$ by stitching copies of $M'$ according to the group $G$. Consider the extension $s: M'' \to \mathbb{R}^2$.

Step 2. Show $s: M'' \to \mathbb{R}^2$ does not degenerate and maintains the orientation of at-least one triangle.

Step 3. If $s$ does not degenerate and maintains orientation of a triangle, it will also not degenerate nor flip orientation of any neighbor triangle.

Step 4. If $s: M'' \to \mathbb{R}^2$ maintains orientation of all triangles it is a homeomorphism. Consequently $f: M \to \mathcal{N}$ is a homeomorphism.
Homeomorphism

- **Step 1.** Build a branched cover $M''$ to $M$ by stitching copies of $M'$ according to the group $G$. Consider the extension $s: M'' \rightarrow \mathbb{R}^2$. All vertices satisfy the CCP.
Homeomorphism

- **Step 2.** We will show a stronger claim. Every (generic) point in the plane is covered by at-least one positively oriented triangle.
• **Step 3.** If $s$ does not degenerate and maintains orientation of a triangle, it will also not degenerate nor flip any neighbor triangle.
Homeomorphism

- **Step 4.** If $s: M'' \to \mathbb{R}^2$ maintains orientation of all triangles it is a homeomorphism. Consequently $f: M \to \mathcal{N}$ is a homeomorphism.

- Repeat winding number argument but now we know that all triangles are positively oriented.
Comparison of CCM
Variational principle

• When $w_{ij} = w_{ji}$ there exists a variational form

$$\min \frac{1}{2} \sum_{e_{ij}} w_{ij} (u_j - u_i)^2$$

s.t. boundary conditions

• This energy is called discrete Dirichlet energy, $E_D(u)$.

• A popular choice of weights comes from asking that $E_D(u) = \int_M |\nabla f|$. 

• These weights are called cotan weights and $w_{ij} = \cot \alpha_{ij} + \cot \beta_{ij}$.

• The mesh is Delaunay ($\alpha_{ij} + \beta_{ij} < \pi$) iff $w_{ij} > 0$. 
Conformality

• The Dirichlet energy satisfies:
  \[ E_D(u) = E_C(u) + E_A(u) \]
  where \( E_A \) is the area functional summing positive areas of triangles.

• The orbifold Tutte theorem implies that \( E_A(u) = area(\mathcal{N}) \) constant.
• Since the number of point constraints matches the degrees of freedom in conformal map we can ask:
  
  Does \( f : M \rightarrow \mathcal{N} \) converge to a conformal map under refinement of \( M \)?

• **Theorem.** Convergence in \( H^1 \) holds for \( \mathcal{N} \) a triangle orbifold. If \( M \) is Delaunay uniform convergence hold.
Discrete uniformization

[Springborn et al. 08]  Orbifold-Tutte
Back to the discrete mapping problem: we got a solution for up-to 4 landmark constraints.

Discrete extremal quasiconformal maps....?
Open problems

• **Problem.** Can $\mathcal{F}$ be enlarged?

• I am not aware of such result.

• **Problem.** Can $\mathcal{F}$ be enlarged under extra conditions?

• Several interesting such results. See notes.
Beyond euclidean

• CCM can be generalized to hyperbolic plane.
• Basic results (Tutte, Orbifold Tutte) still holds.
• Allows infinite number of cones.
• **Drawback:** no longer a linear model
Beyond euclidean
Higher dimensions?

- Counter example to Tutte exists. The following example by [Floater, Pham-Trong].
The end

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