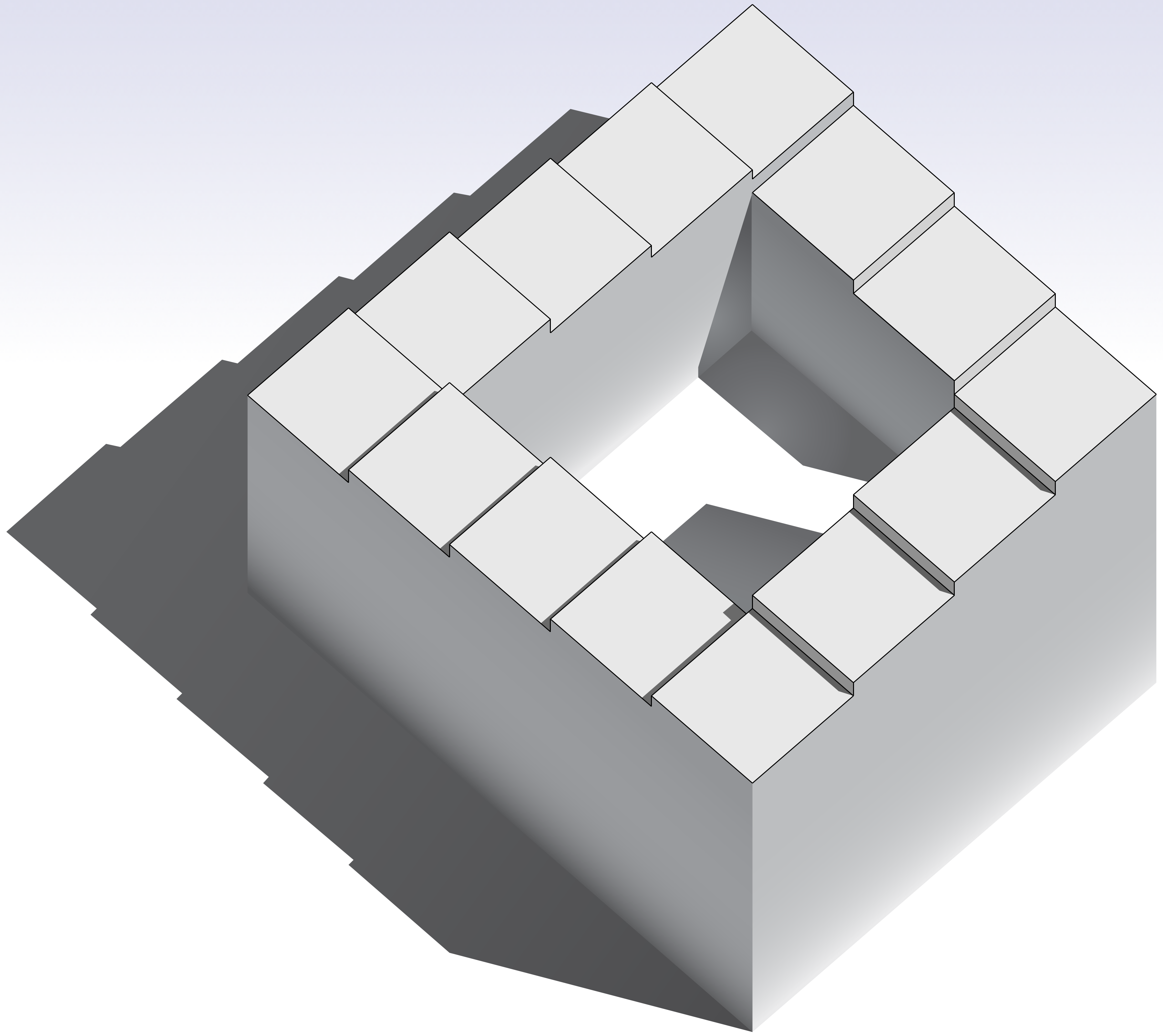
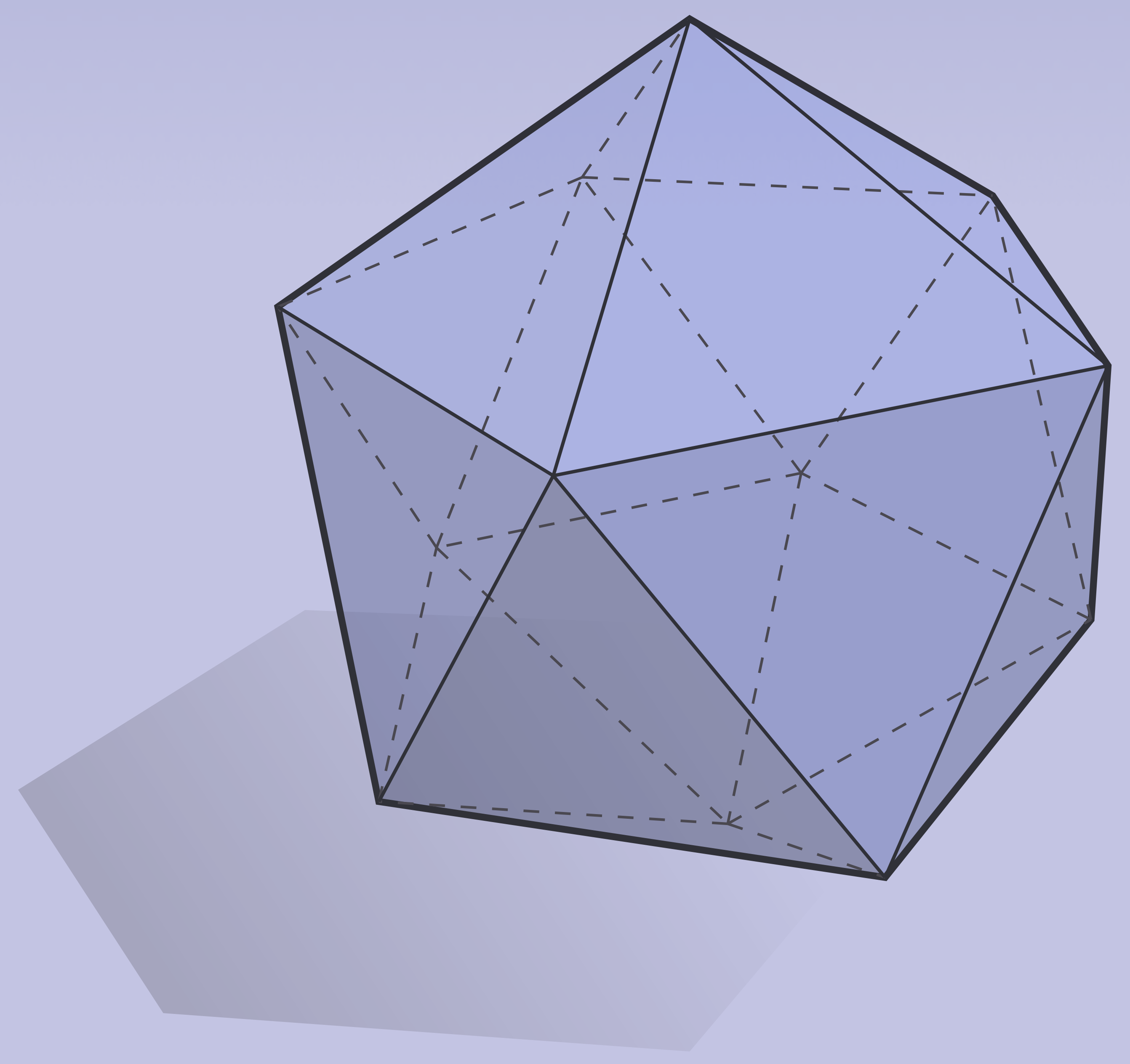


AMS Short Course

Discrete Differential Geometry

Joint Mathematics Meeting • San Diego, CA • January 2018

Discrete Conformal Geometry



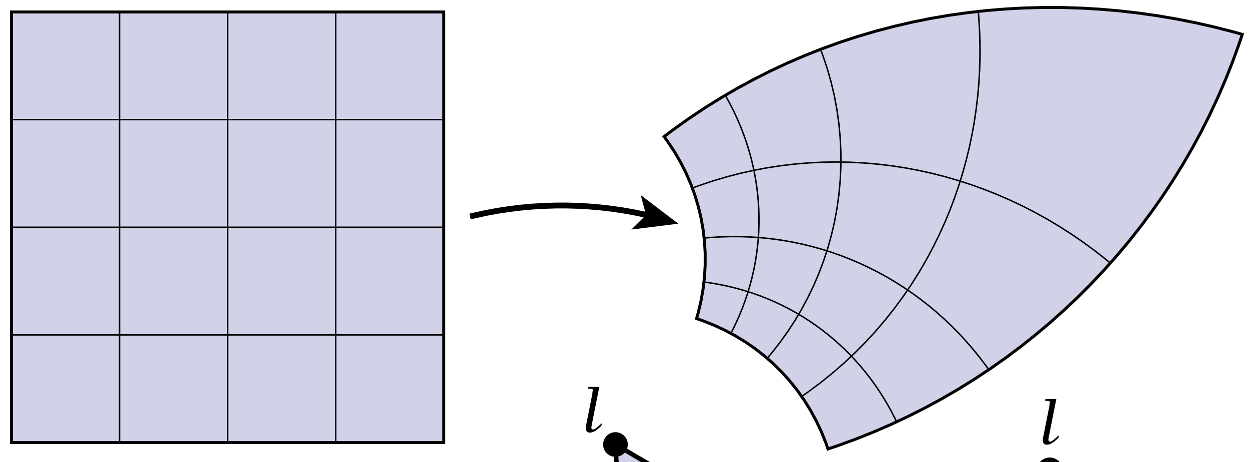
AMS Short Course

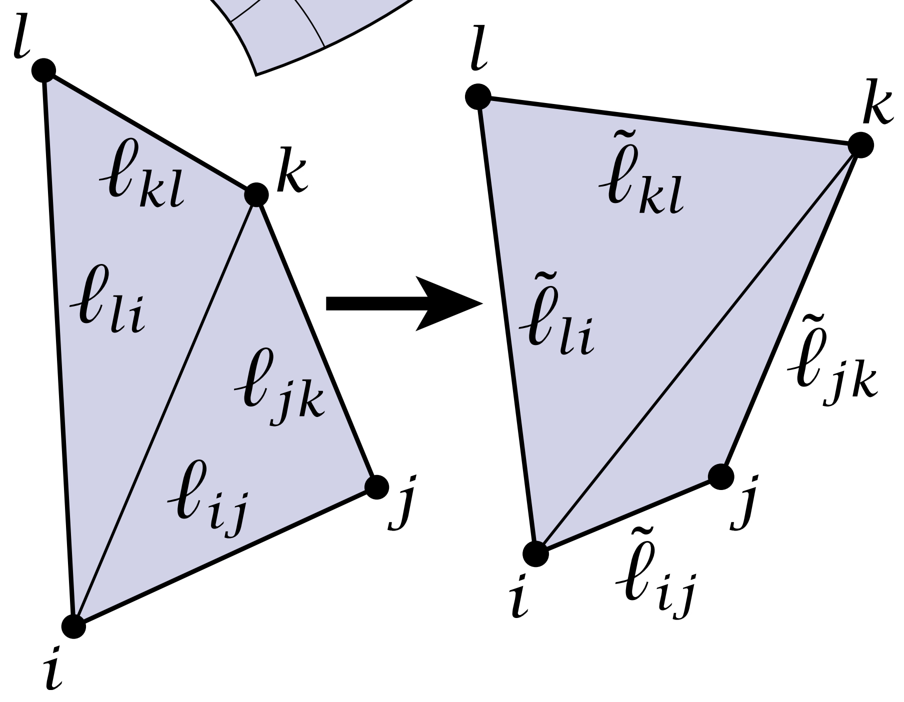
Discrete Differential Geometry

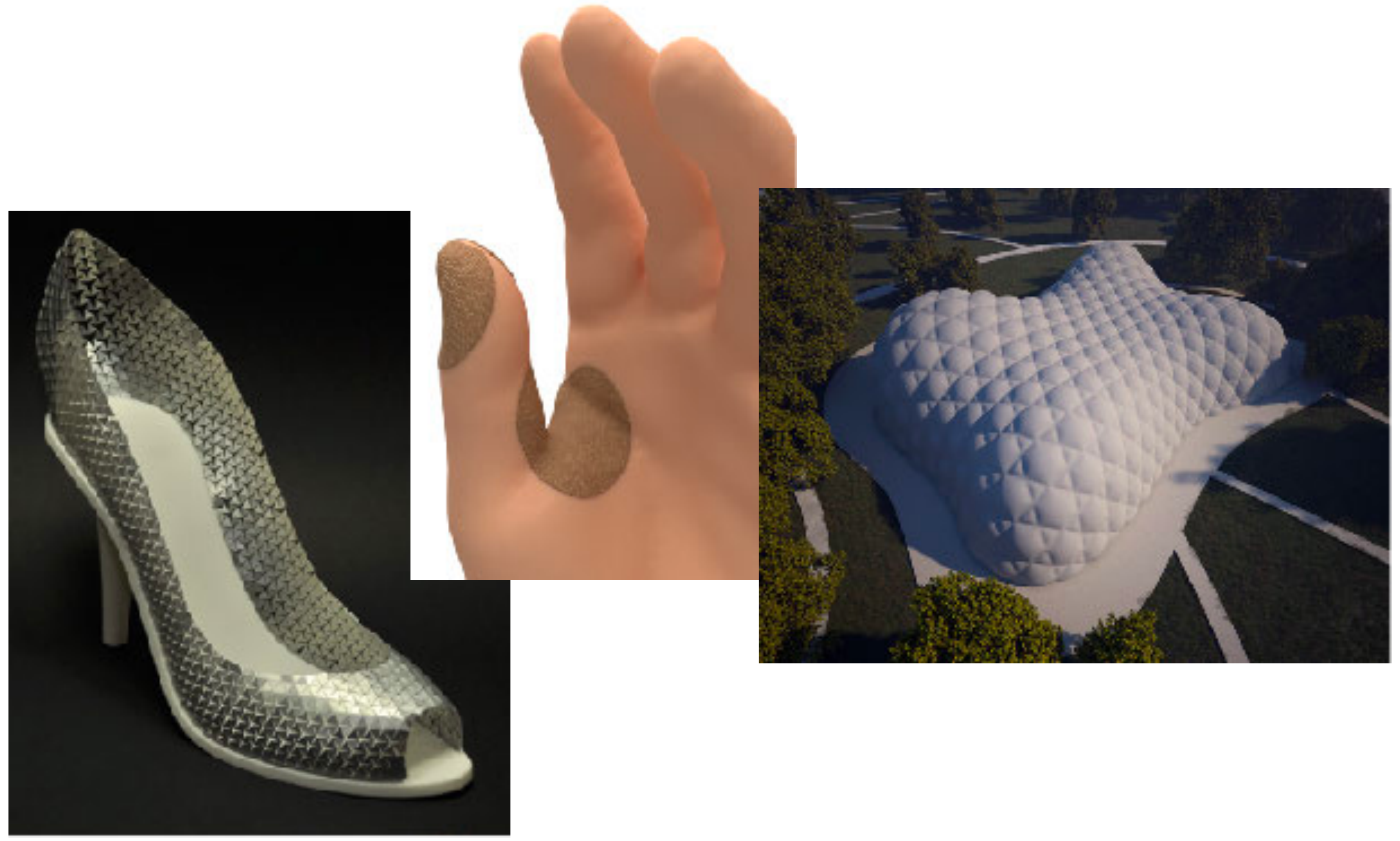
Joint Mathematics Meeting • San Diego, CA • January 2018

Conformal Geometry — Overview

- Schedule:
  - Part I: Theory
  - *(lunch)*
  - Part II: Applications







- Extremely rich field; barely enough time to scratch the surface (not even quads!)
- Focus on one "small problem" of discretization
  - illustrate **The Game** of DDG
  - previously encountered *no free lunch* situation (curvature)
  - this time will see another common theme: *rigidity*

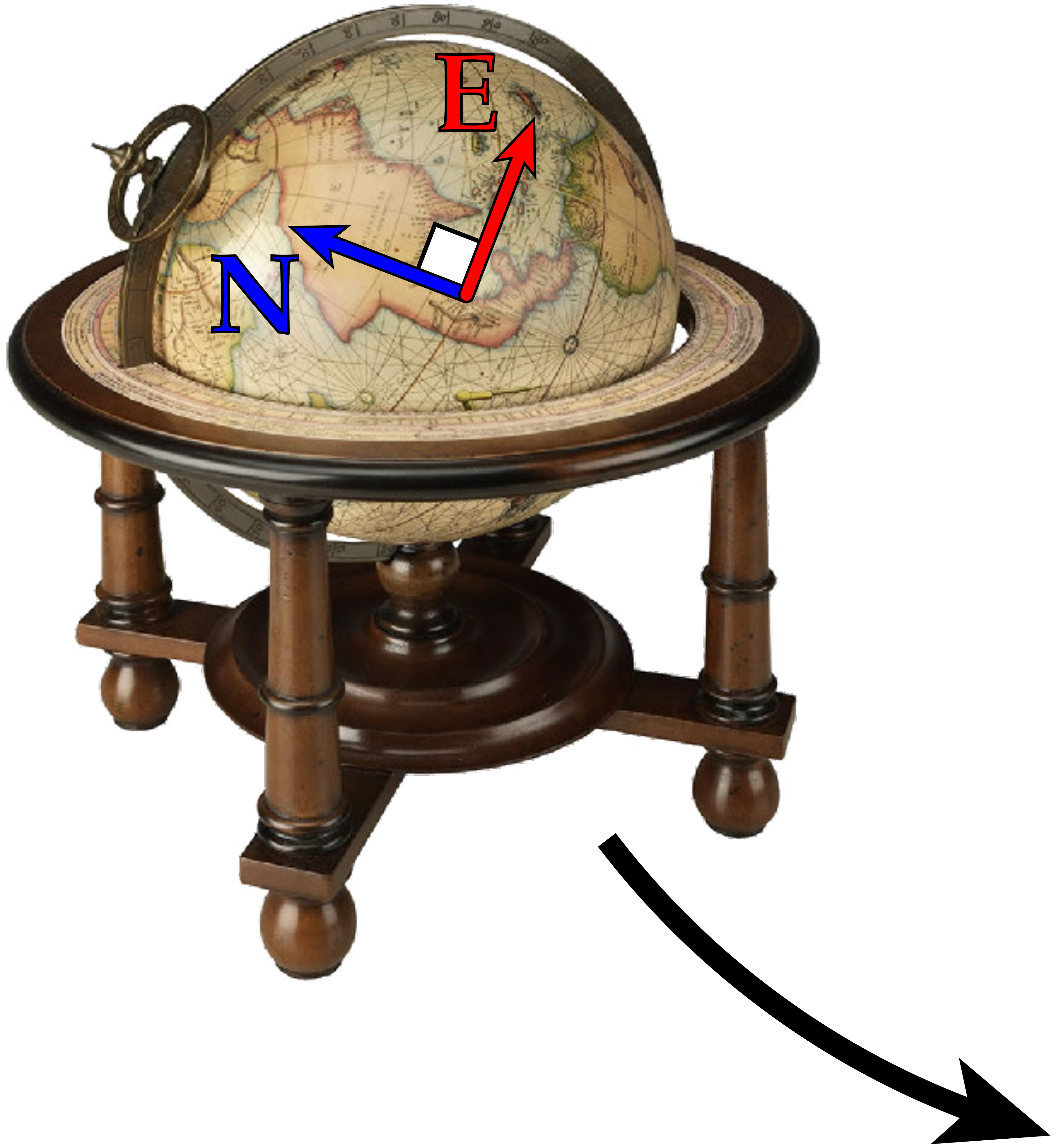
Motivation: Mapmaking Problem

- How do you make a flat map of the round globe?
- Hard to do! Like trying to flatten an orange peel…

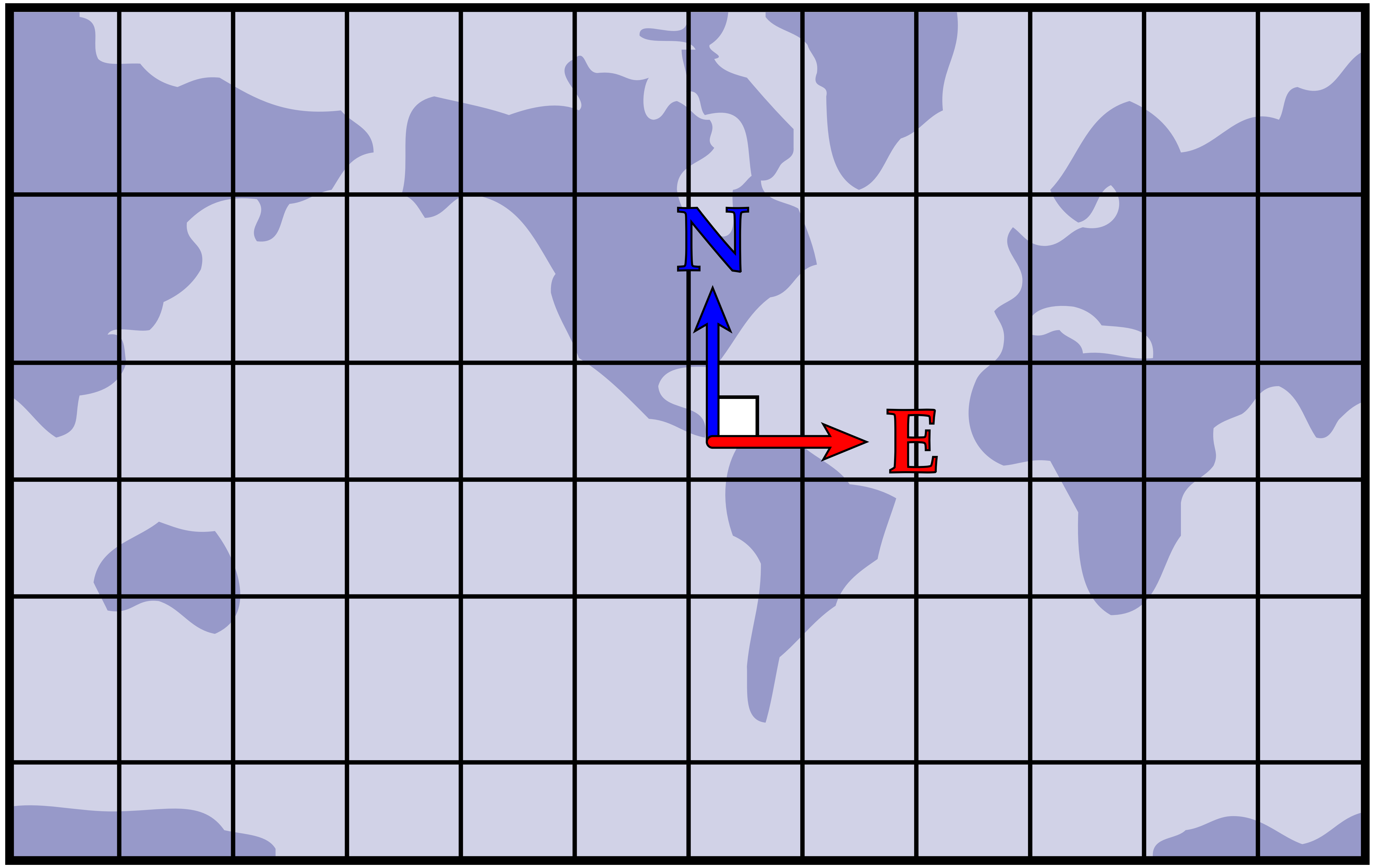
Impossible without some kind of distortion and/or cutting.

Conformal Mapmaking

- Amazing fact: can always make a map that exactly preserves **angles**.

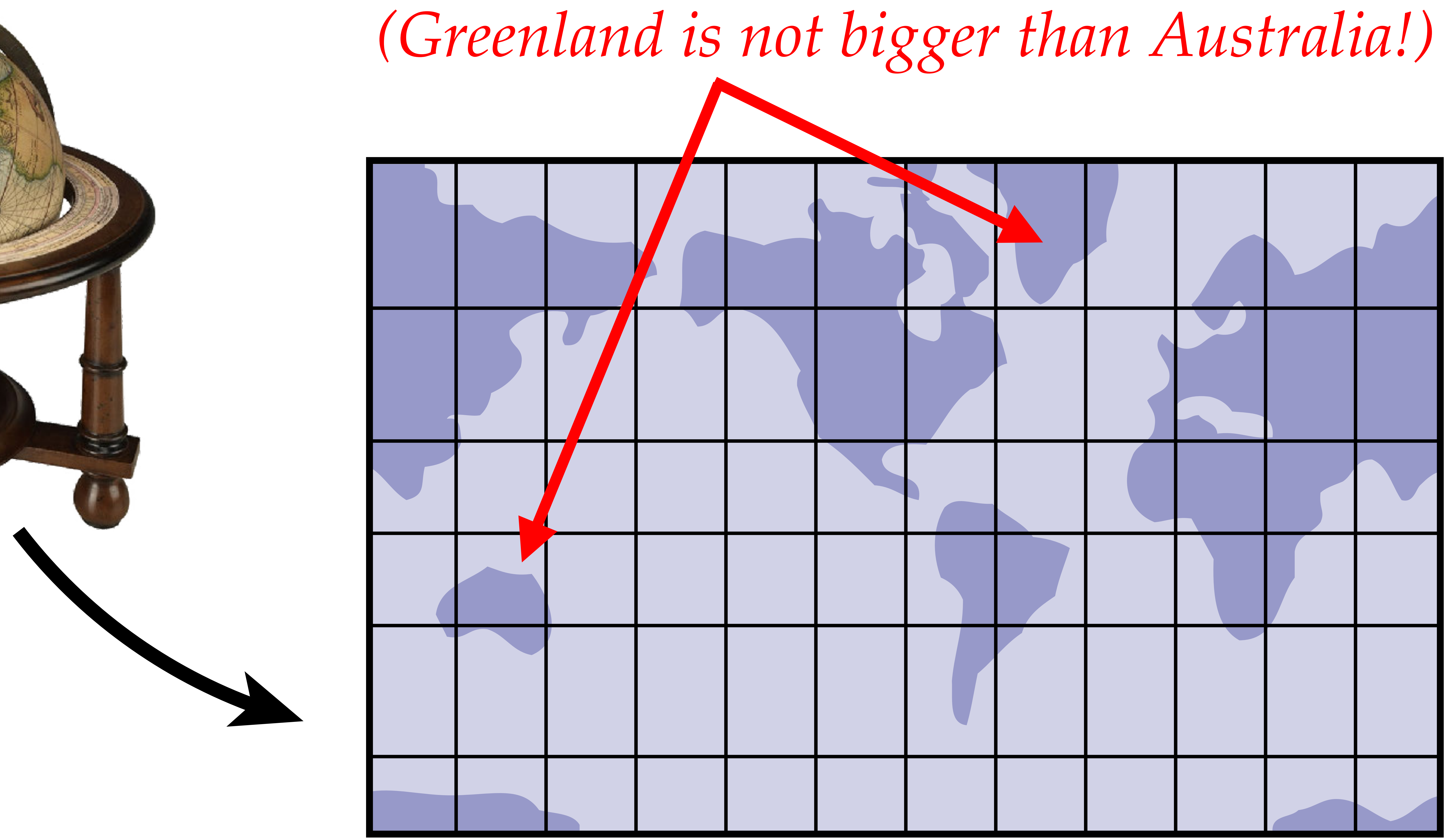


(Very useful for navigation!)



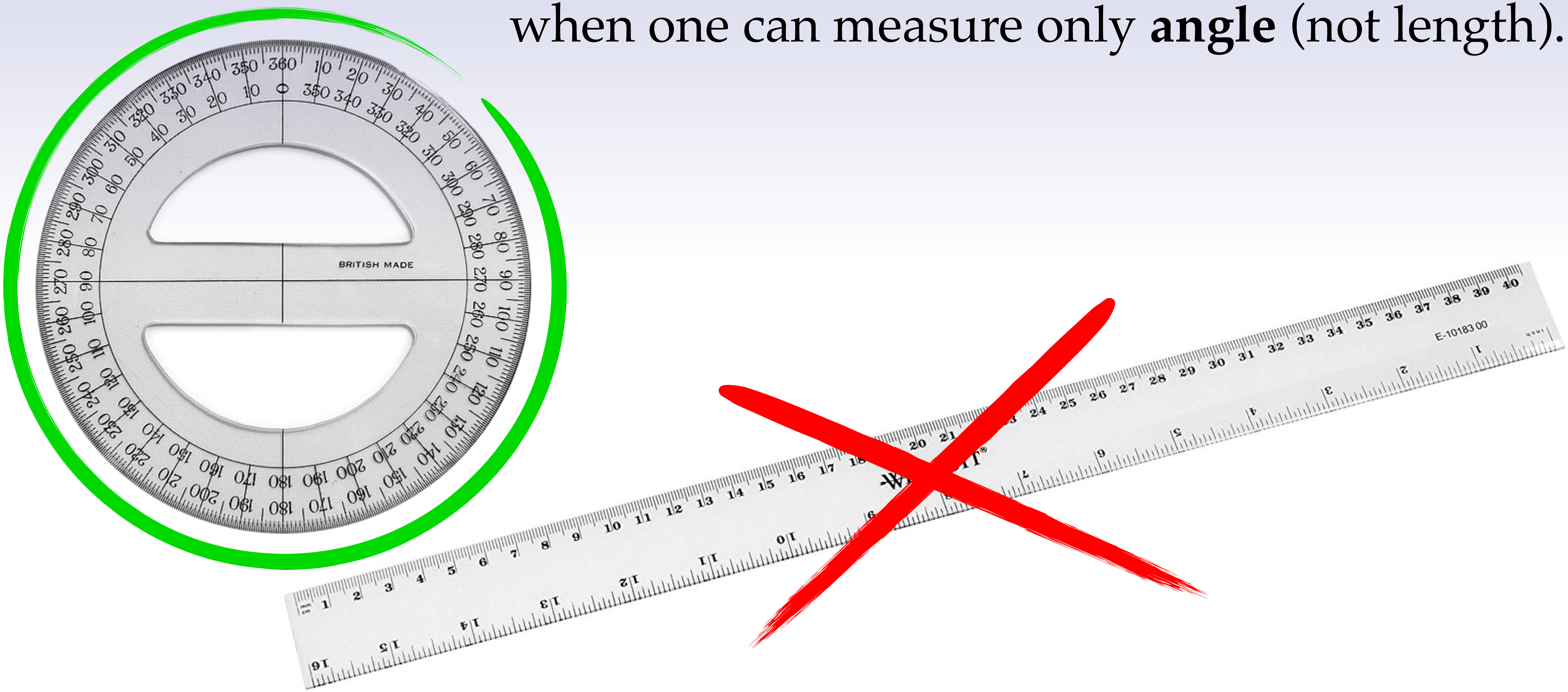
Conformal Mapmaking

- However, **areas** may be badly distorted…



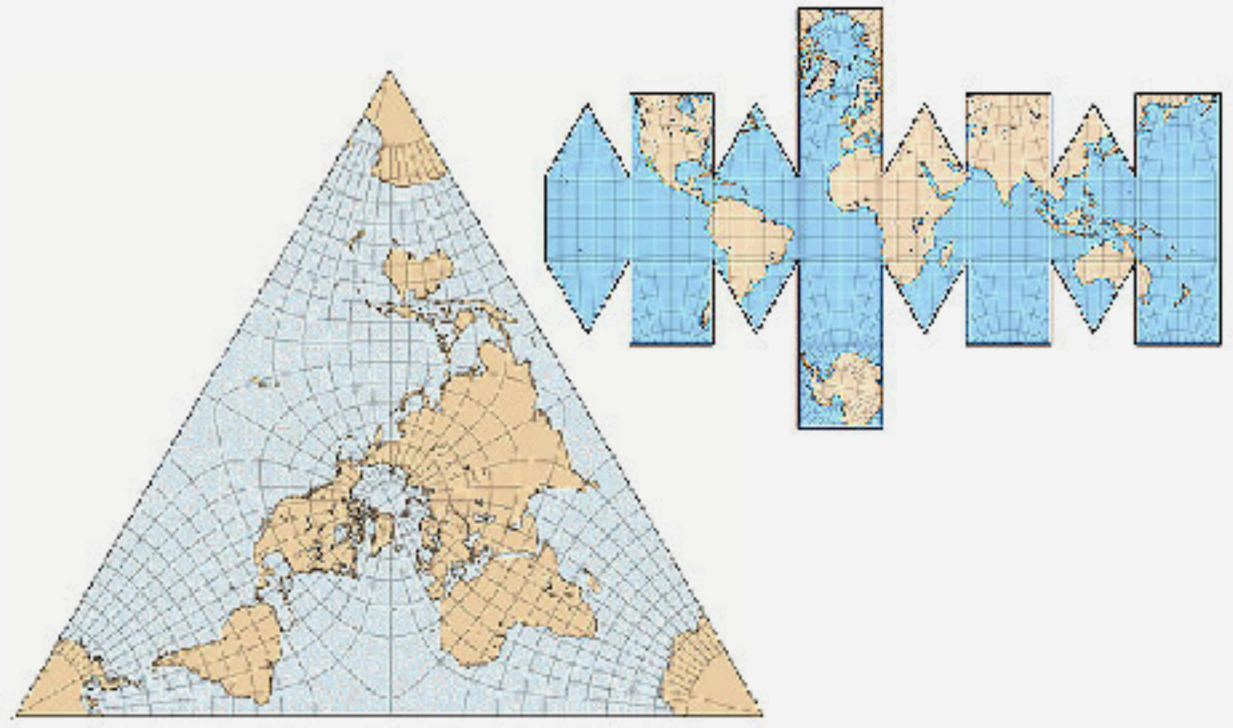
Conformal Geometry

More broadly, *conformal geometry* is the study of shape when one can measure only **angle** (not length).

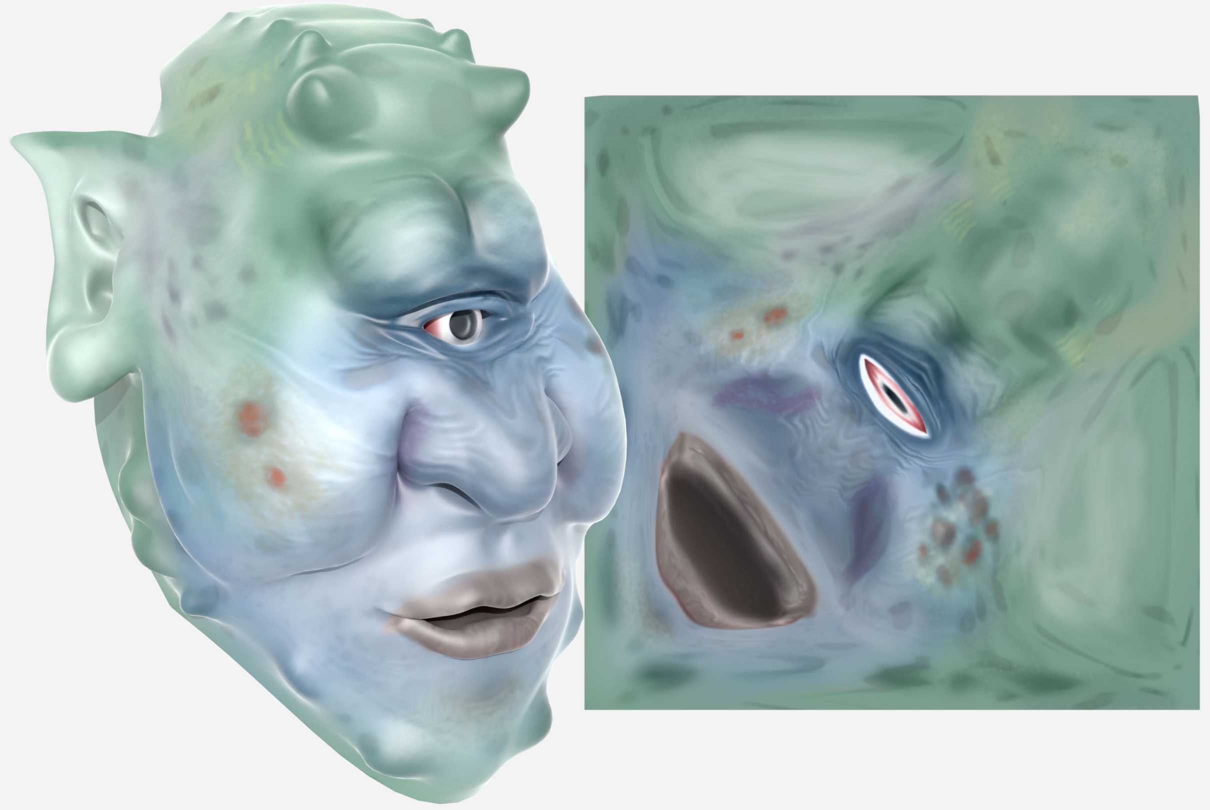


Applications of Conformal Geometry Processing

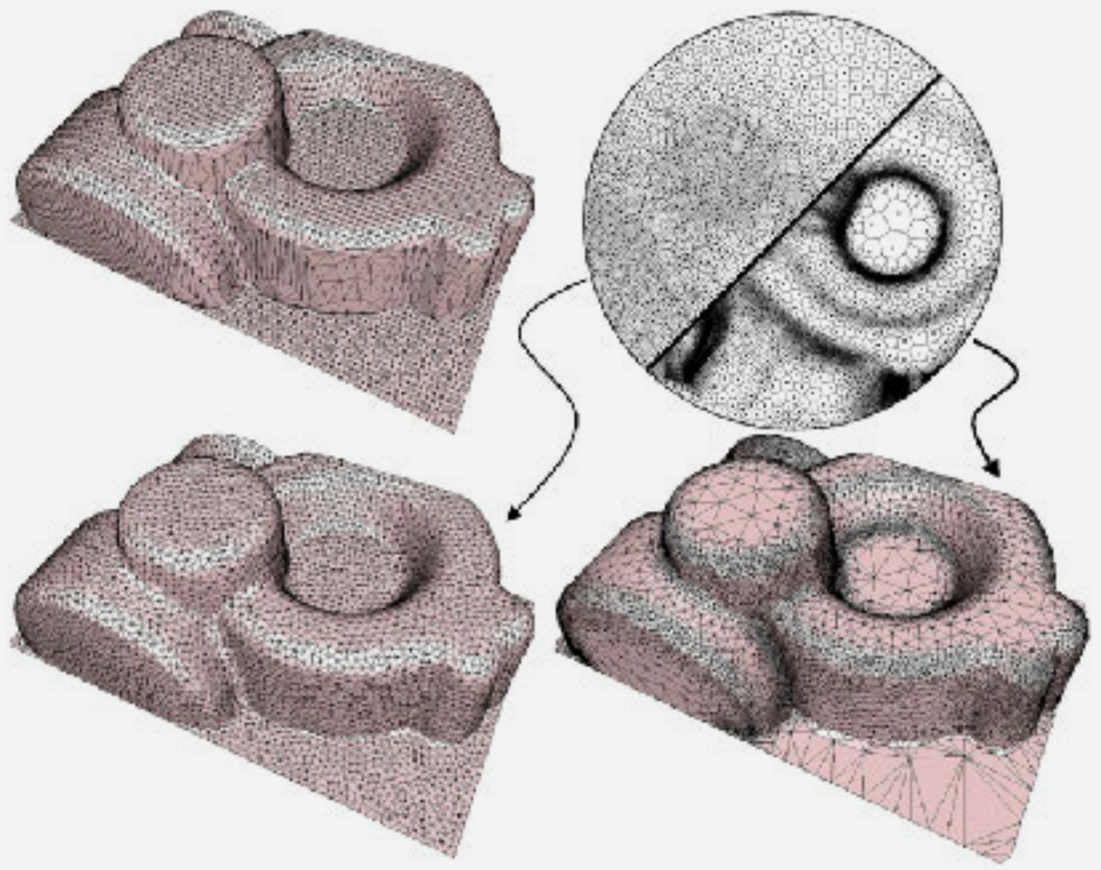
Basic building block for *many* applications…



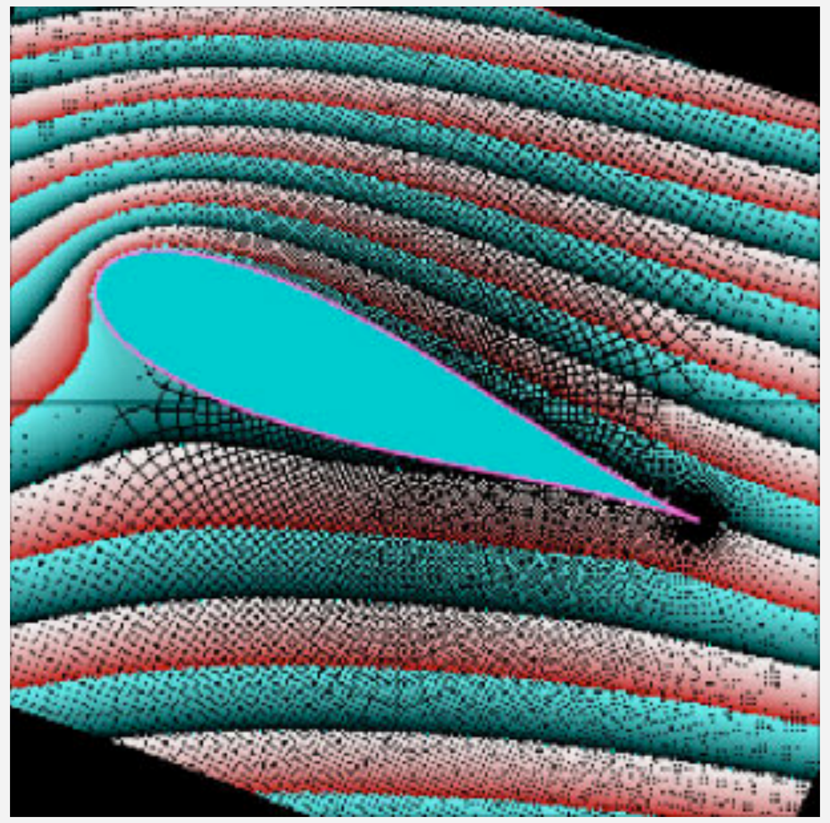
CARTOGRAPHY



TEXTURE MAPPING

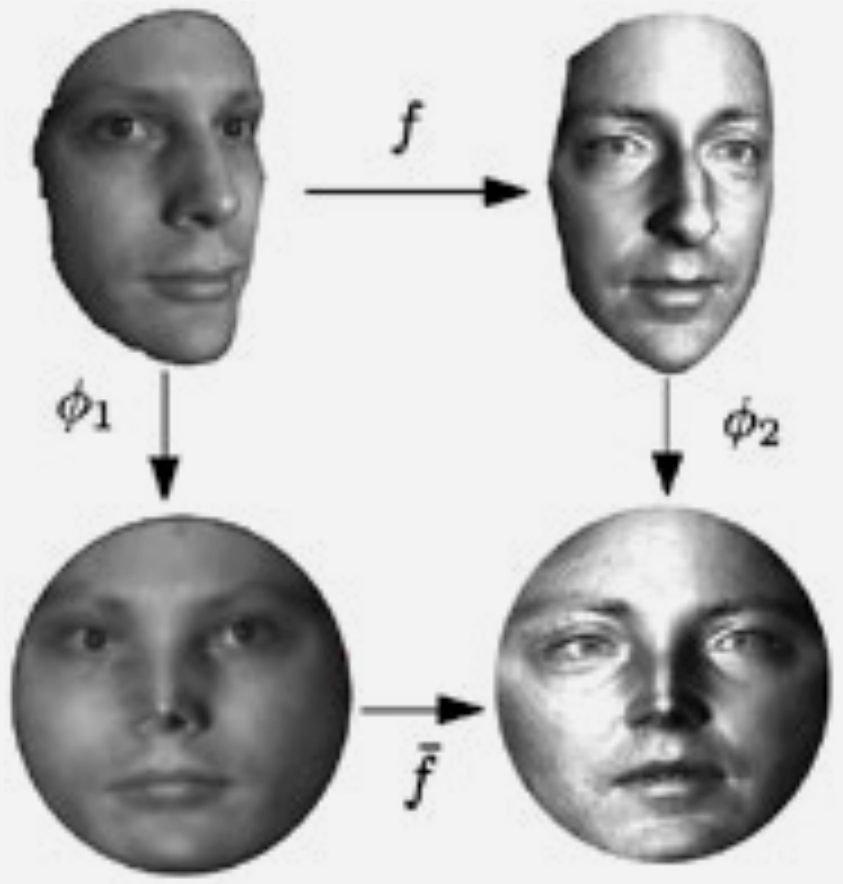


REMESHING

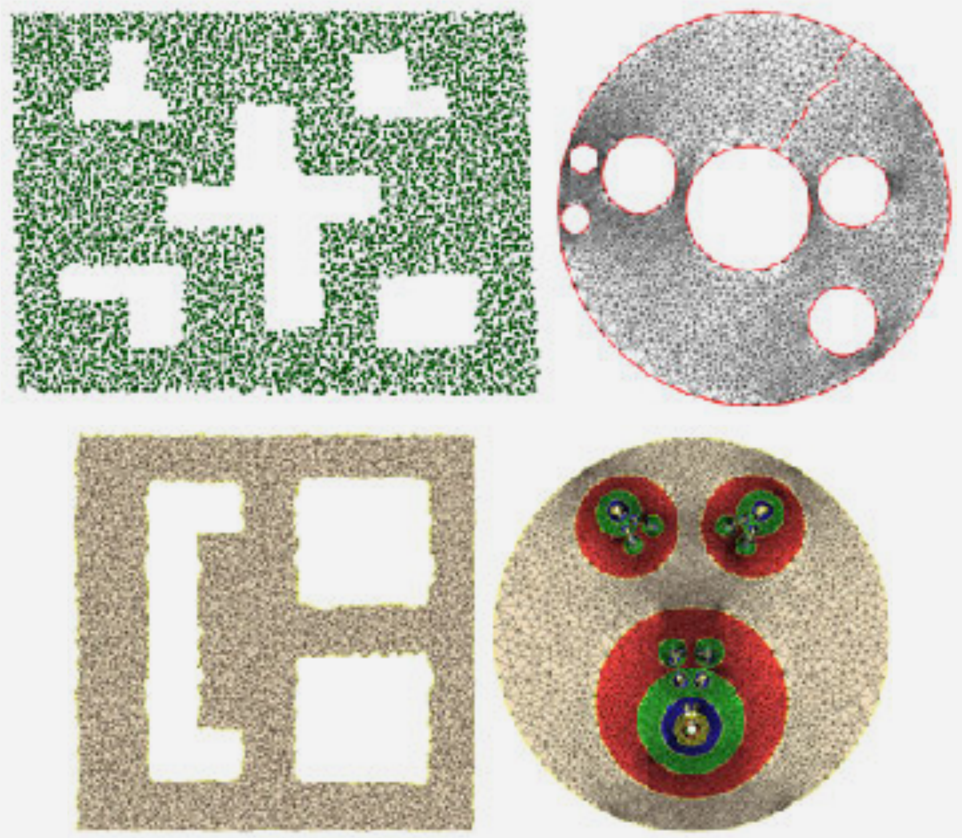


SIMULATION

3D FABRICATION



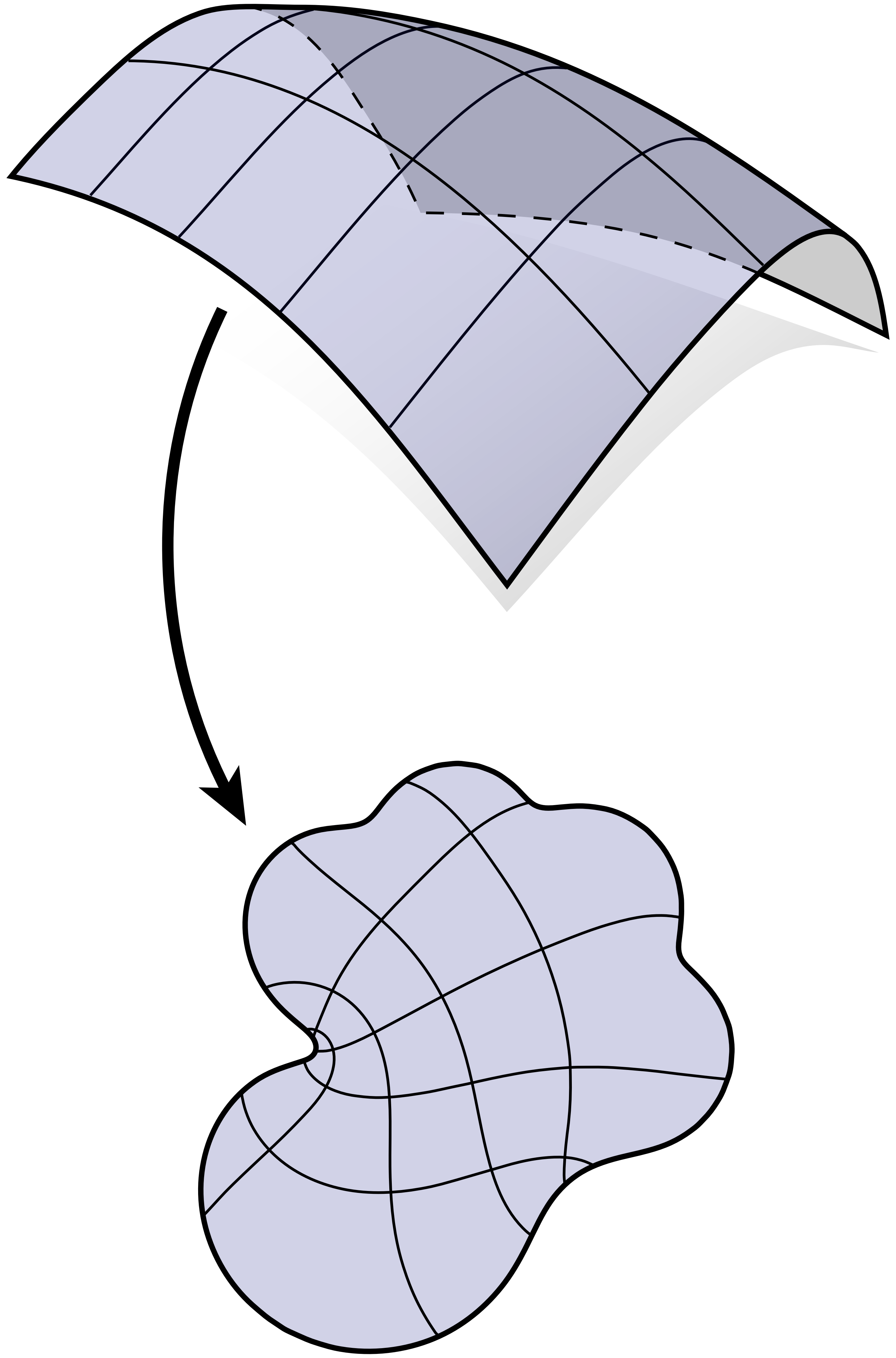
SHAPE ANALYSIS



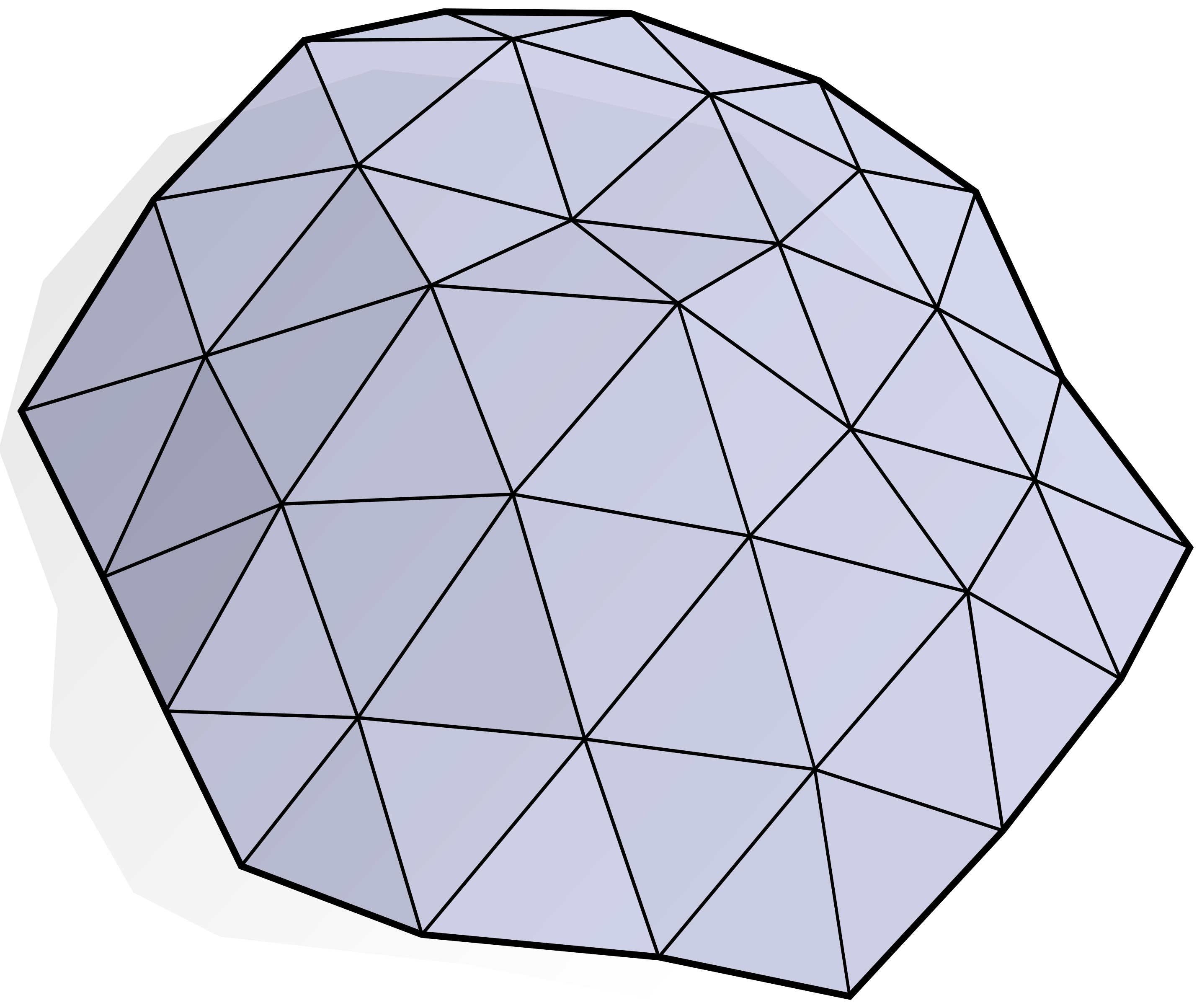
SENSOR NETWORKS

A Small Problem in Discretization

- Given a simplicial disk with a discrete metric, how do we **define** a *conformal flattening*, i.e., a conformal map to the flat Euclidean plane?
- Many possible characterizations in smooth setting…
- But will encounter issue of *rigidity* / *flexibility*
  - some definitions have **fewer** degrees of freedom than in smooth setting ("too rigid")
  - others have **more** degrees of freedom ("too flexible")
  - one and only one theory is "just right" (conformal equivalence of discrete metrics)
- *Other approaches still provide useful perspective—and algorithms!*



Problem Statement



- An **(abstract) simplicial disk** is a simplicial 2-complex $K = (V,E,F)$ such that the link of every vertex is either a single cycle or path.
- A **discrete metric** on K is an assignment of positive edge lengths $\ell_{ij} > 0$ to each edge ij in E such that the triangle inequality holds in each triangle, *i.e.*,

$$\ell_{ij} + \ell_{jk} \geq \ell_{ki} \quad \forall ijk \in F$$

- (Geometrically, can view (K,ℓ) as a piecewise Euclidean surface obtained by gluing together Euclidean triangles along edges)

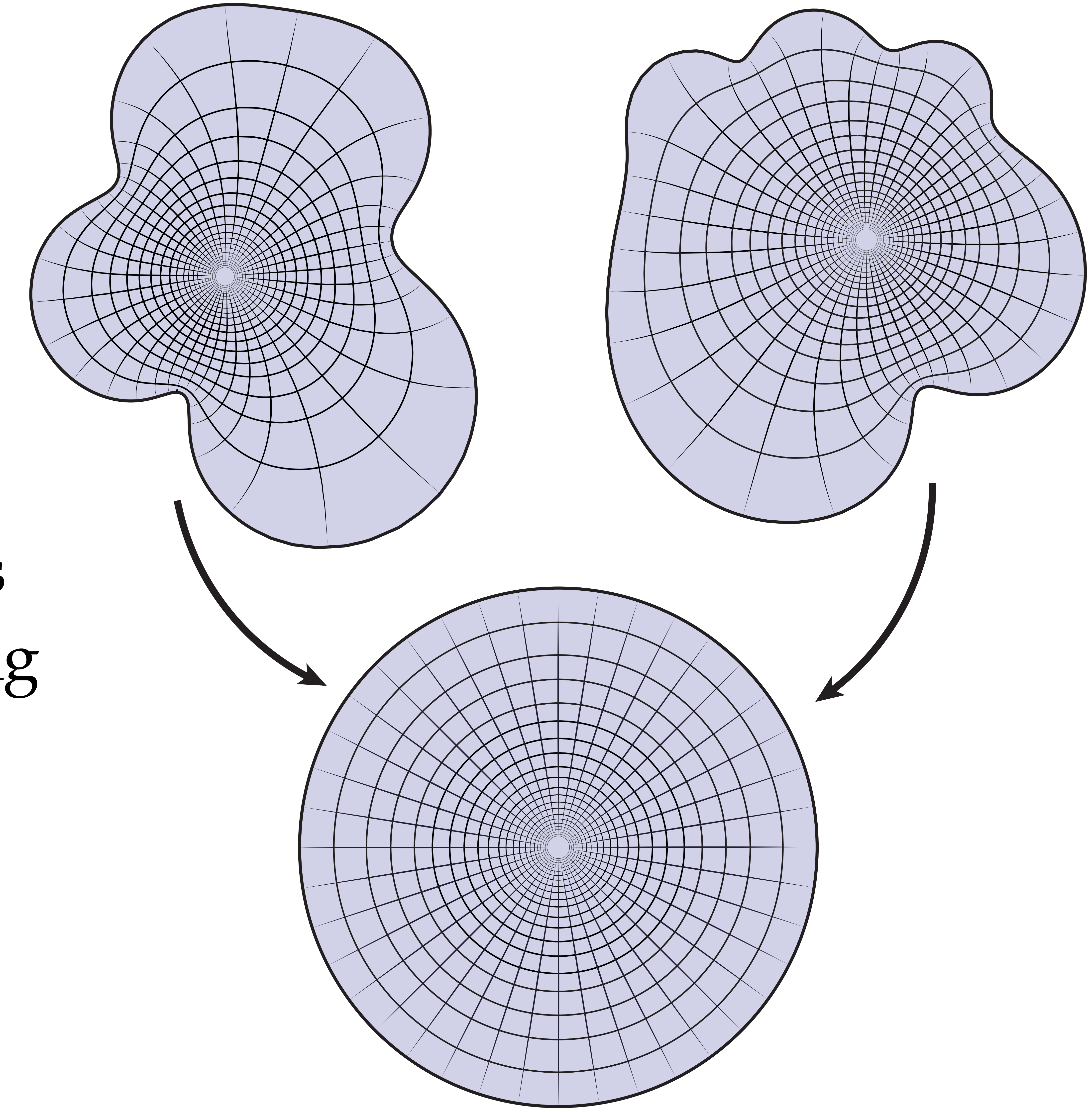
Problem: find a definition for discrete conformal maps $f : V \to \mathbb{R}^2$ that (i) exhibit the same flexibility as smooth conformal maps and (ii) ~~converge under refinement.~~

Flexibility of Conformal Flattening

- In the smooth setting, how flexible are conformal flattenings of a disk?
- **Theorem** (Uniformization). Every Riemannian disk (M,g) can be conformally mapped to the unit disk in the plane.
- **Fact.** The space of conformal disk flattenings can be parameterized by a real function along on the boundary specifying either

(i) the scale factor along the boundary, or

(ii) the target curvature of the boundary.



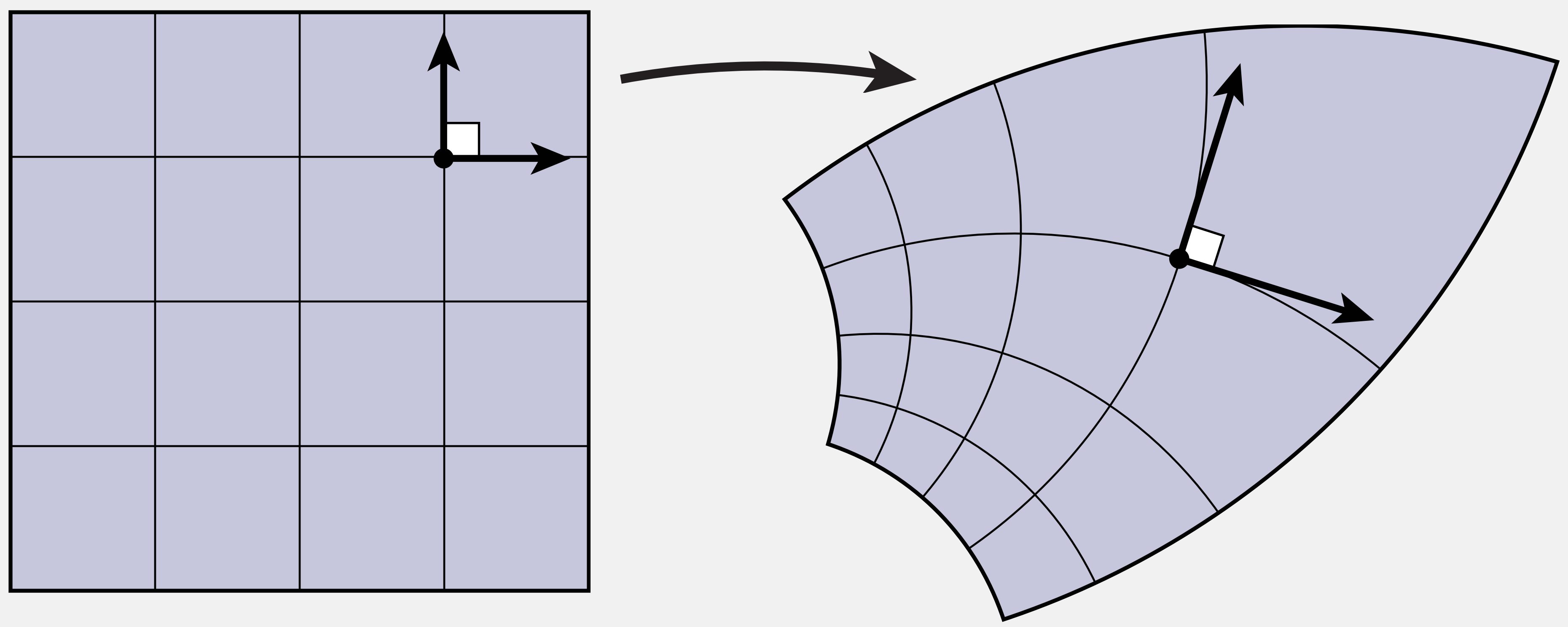
The Game of DDG

Recall our basic approach to finding new discrete definitions:

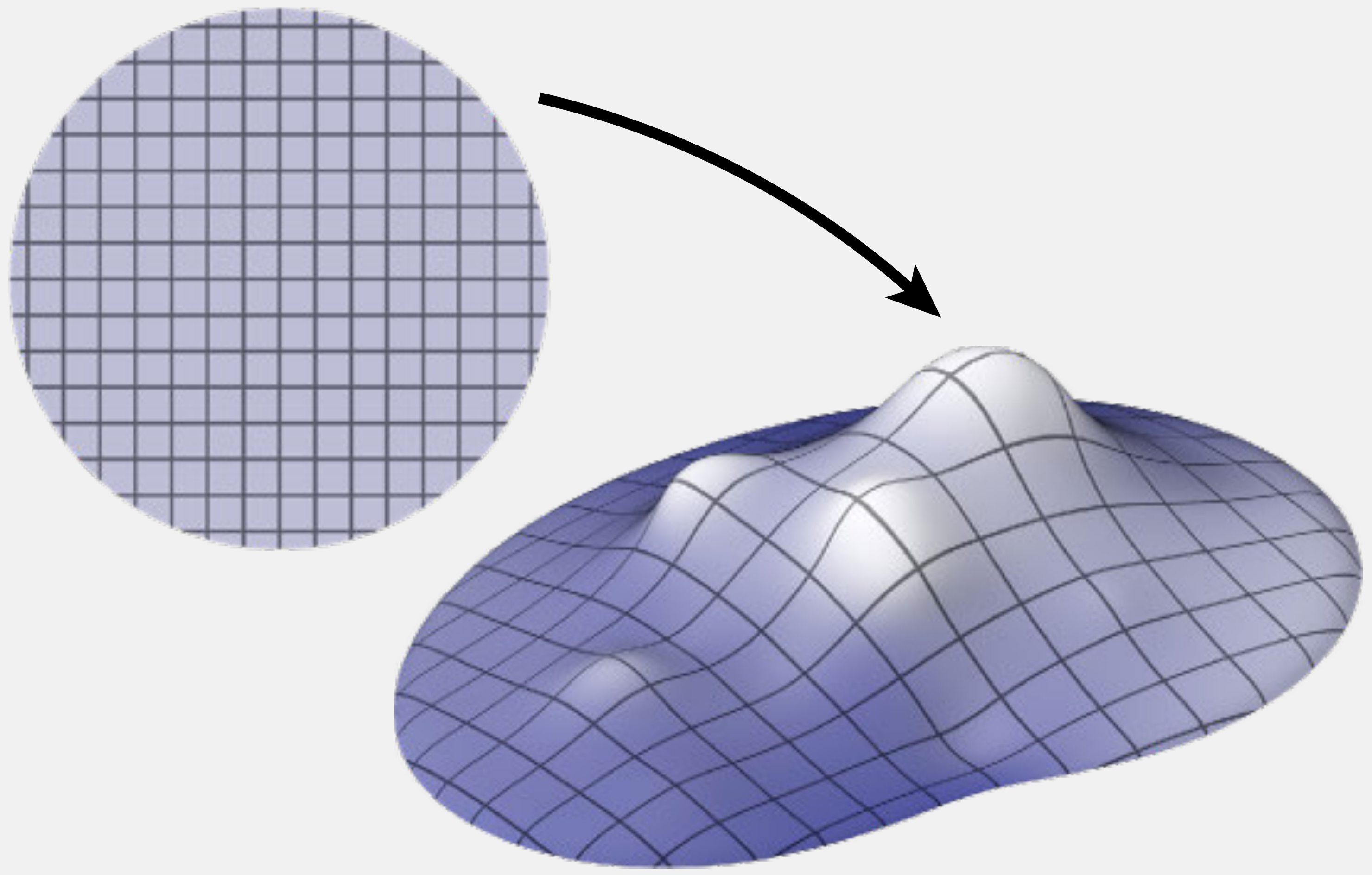
The Game

1. Write down several **equivalent** definitions in the smooth setting.
2. (Try to) apply each smooth definition to a given discrete object.
3. Check which properties of the smooth object are preserved.

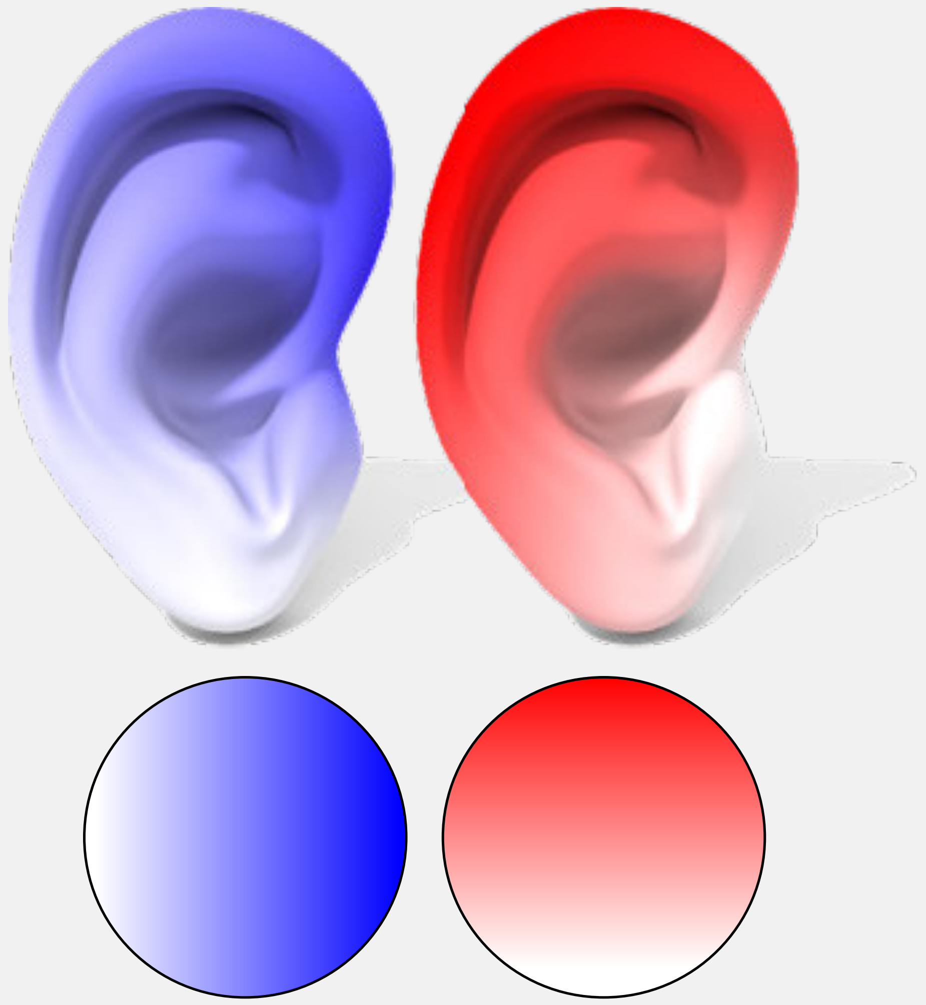
(Some) Smooth Characterizations of Conformal Maps



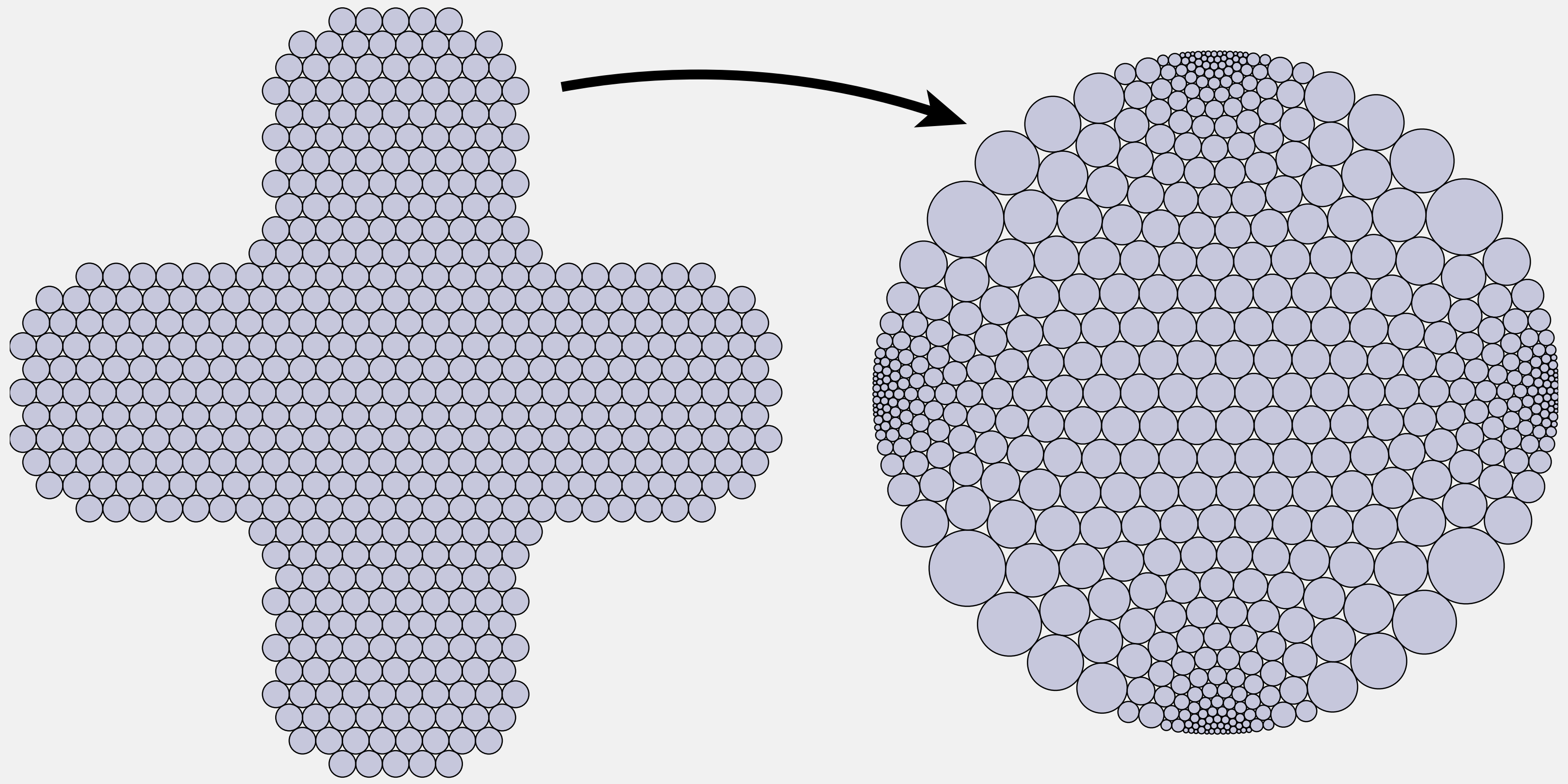
angle preservation



metric rescaling



conjugate harmonic functions



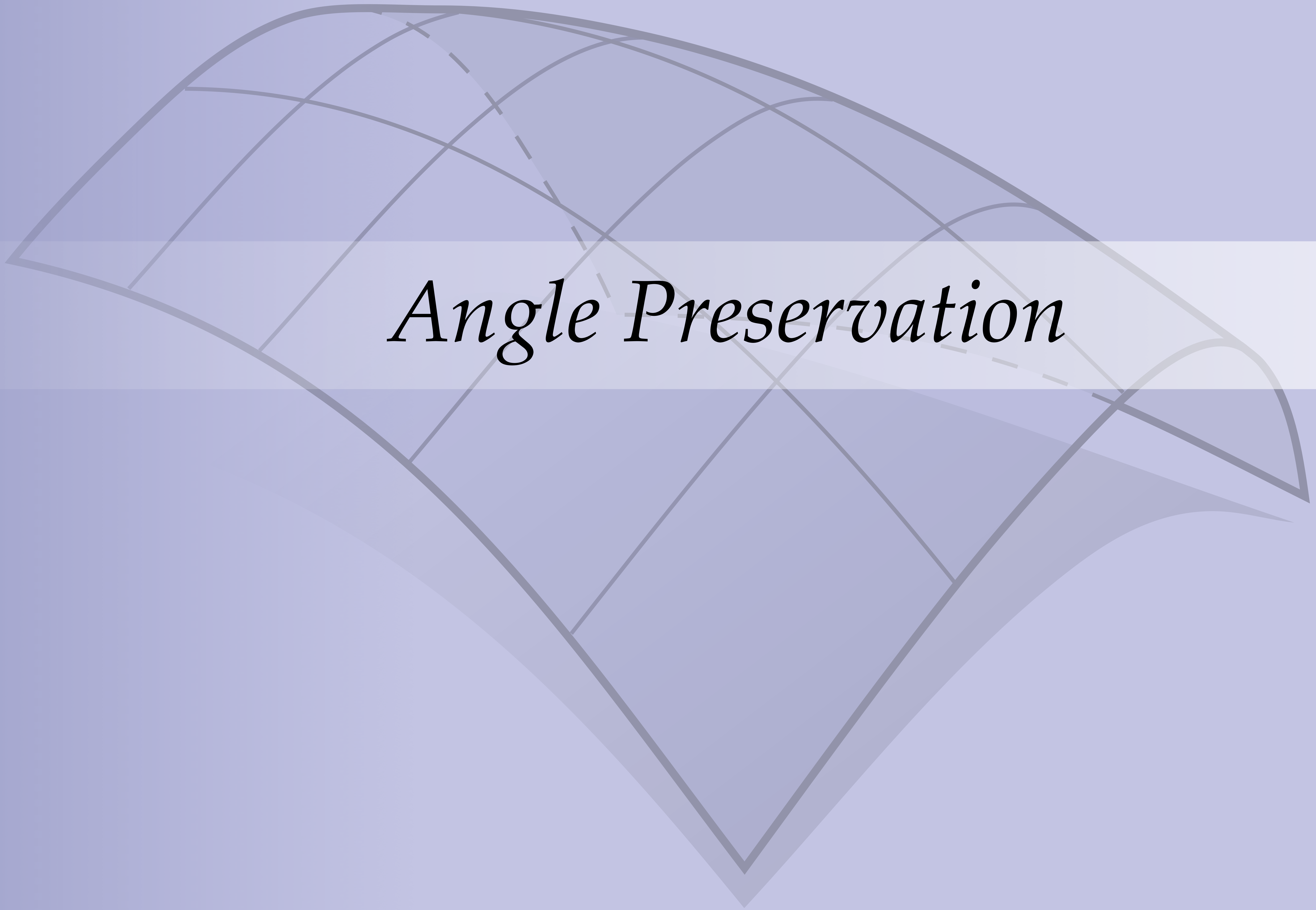
preservation of circles

critical points of Dirichlet energy

(Some) Approaches to Discretization

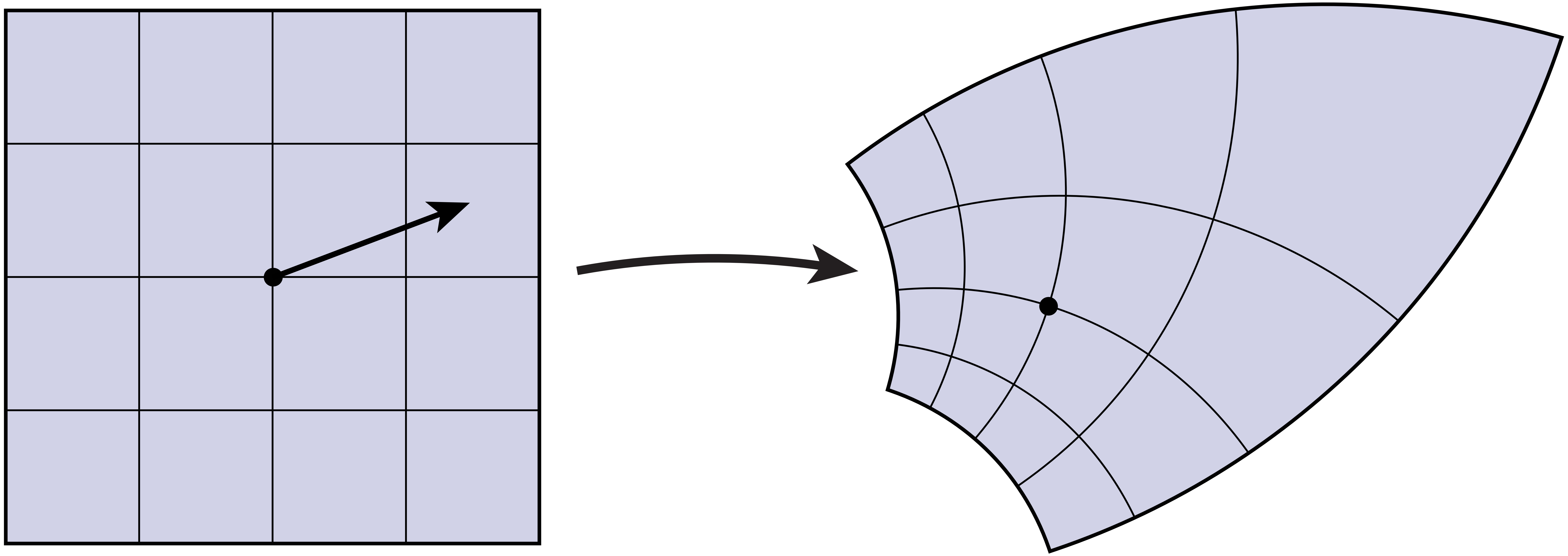
Characterization	Discretization / Algorithm
Cauchy-Riemann	*least square conformal maps (LSCM)*
Dirichlet energy	*discrete conformal parameterization (DCP)* *genus zero surface conformal mapping (GZ)*
angle preservation	*angle based flattening (ABF)*
circle preservation	*circle packing* *circle patterns (CP)*
metric rescaling	*conformal prescription with metric scaling (CPMS)* ***conformal equivalence of triangle meshes (CETM)***
conjugate harmonic	*boundary first flattening (BFF)*

Angle Preservation



Angle Preservation

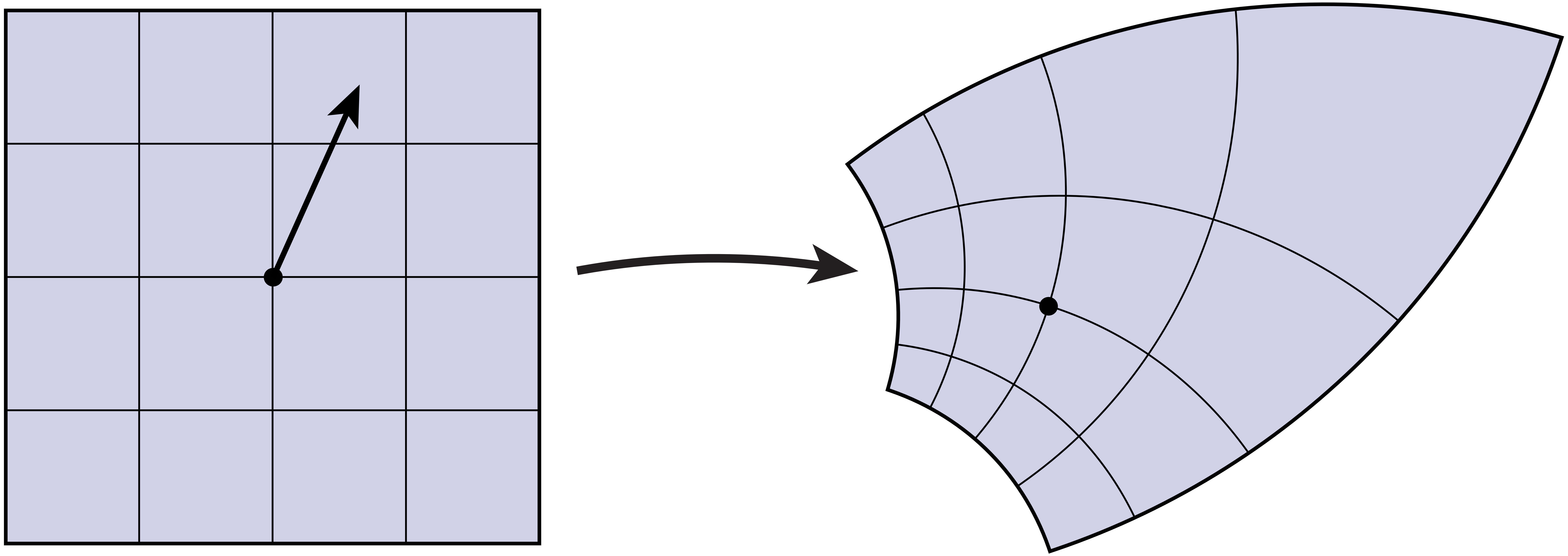
Let's start with an elementary definition…



"Conformal maps preserve angles"

Angle Preservation

Let's start with an elementary definition…



"Conformal maps preserve angles"

Angle Preservation

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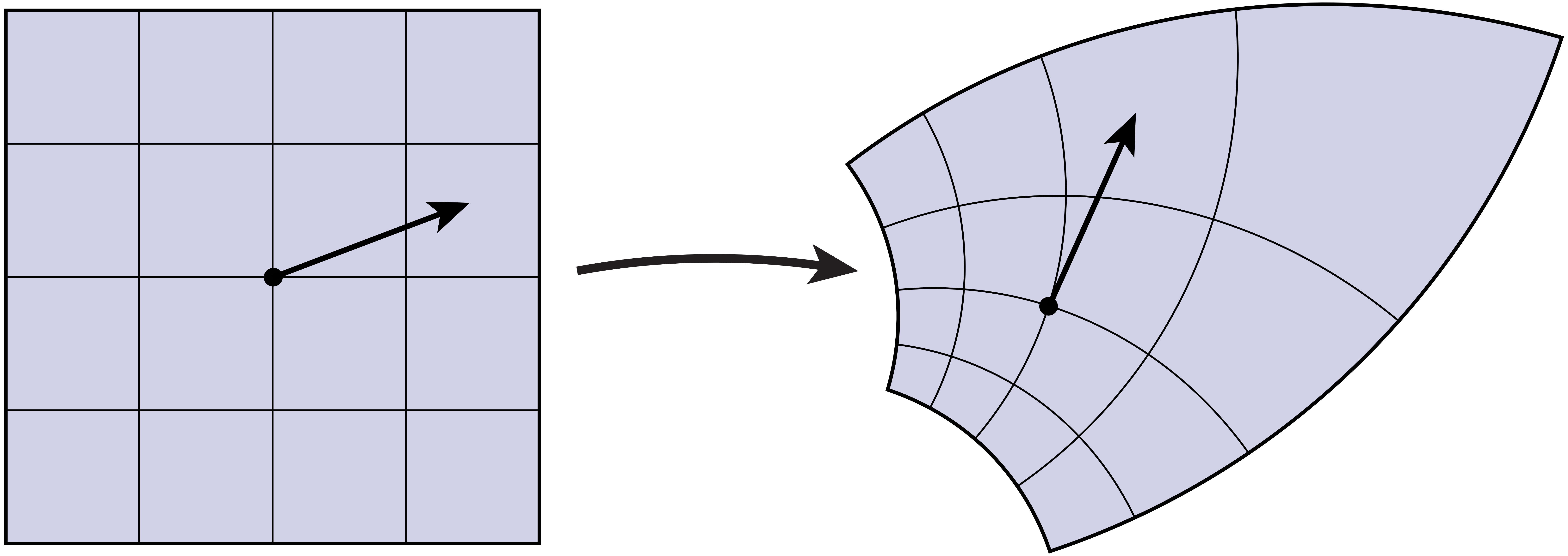
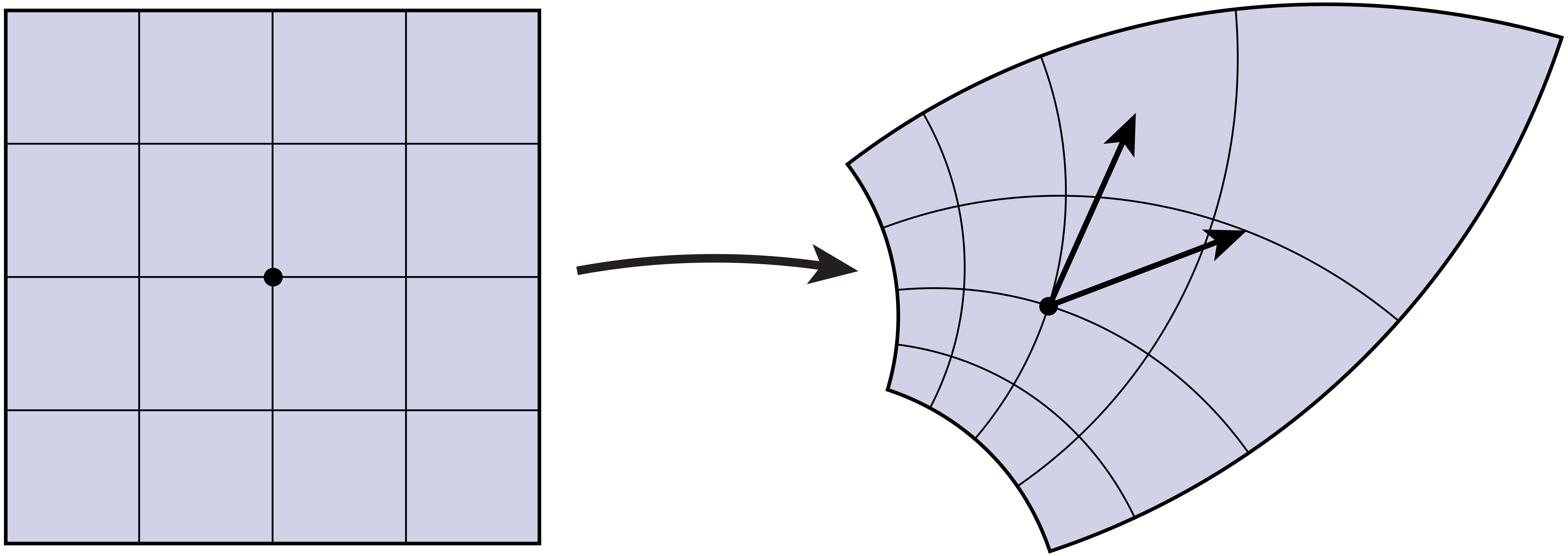


"Conformal maps preserve angles"

Angle Preservation

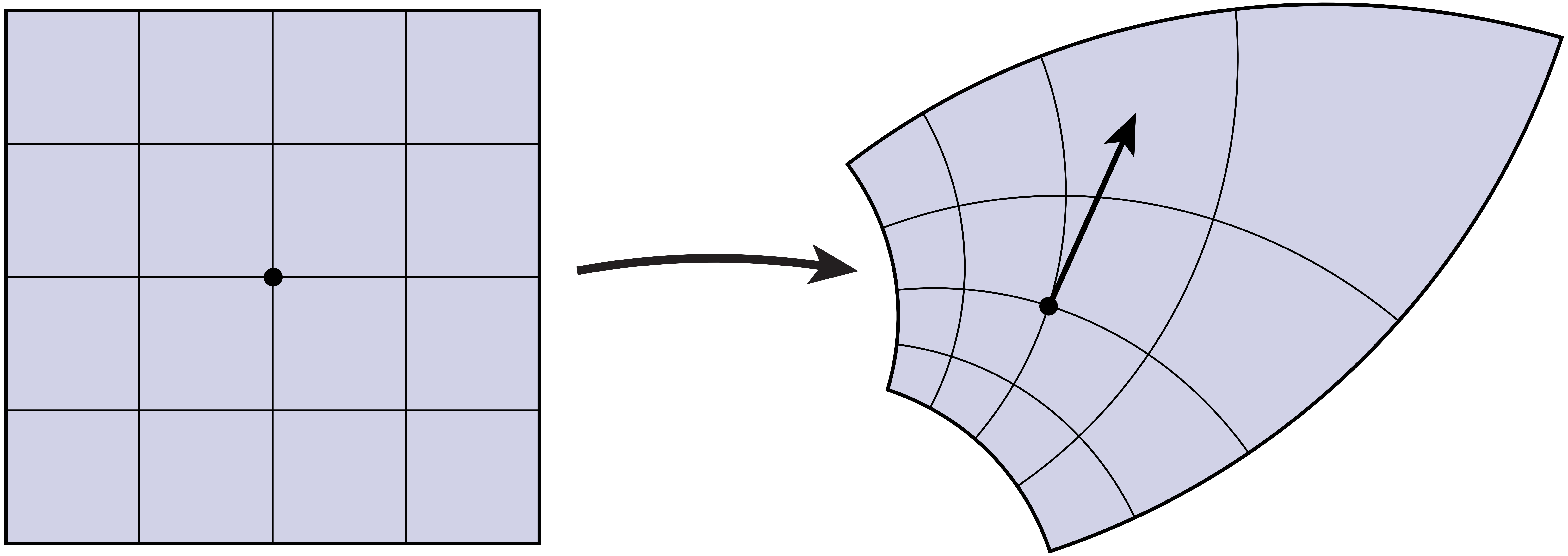
Let's start with an elementary definition…



"Conformal maps preserve angles"

Angle Preservation

Let's start with an elementary definition…



"Conformal maps preserve angles"

Angle Preservation

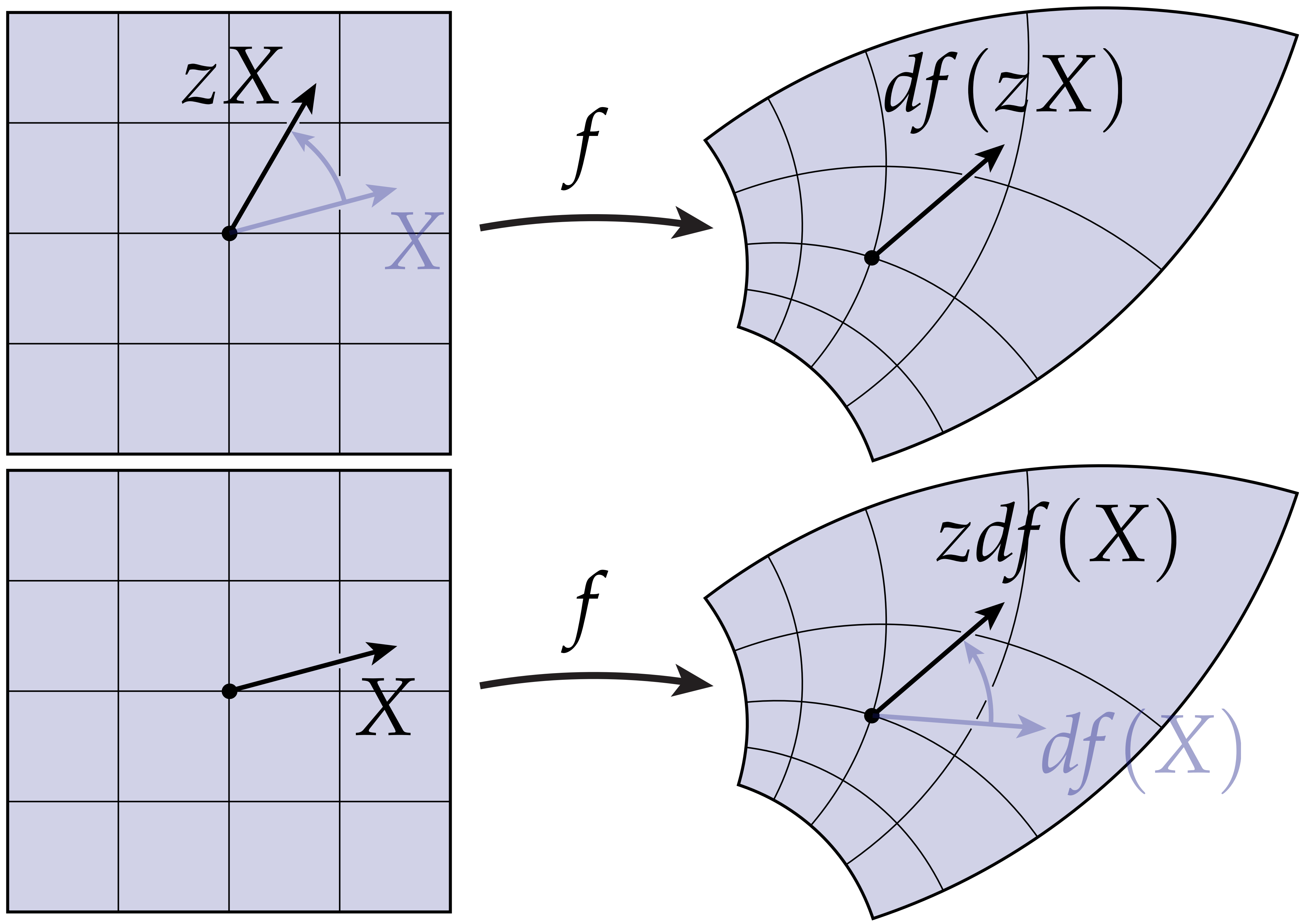
A map $f : \mathbb{C} \supset U \to \mathbb{C}$ is **holomorphic** if at all points $p \in U$

$$df_p(zX) = zdf_p(X)$$

for all unit complex numbers z and tangent vectors X.

It is **conformal** if it is also immersion, *i.e.*, if

$$df(X) = 0 \iff X = 0.$$

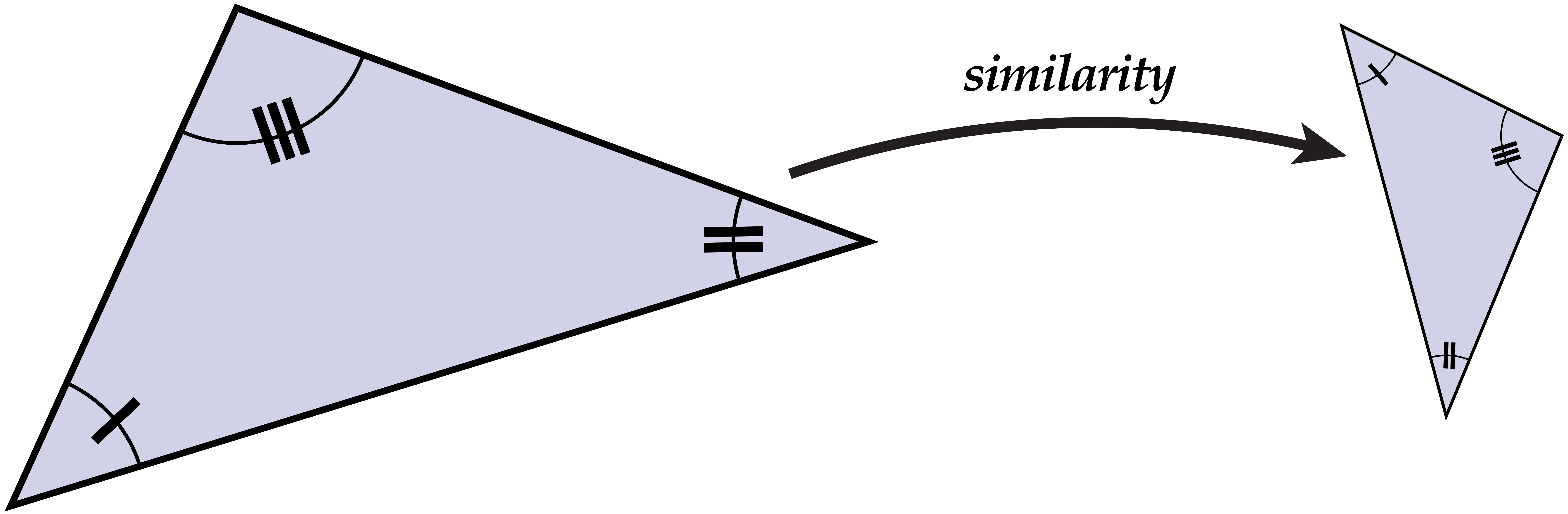


Intuitively: f preserves angles & orientation.

Discrete Angle Preservation

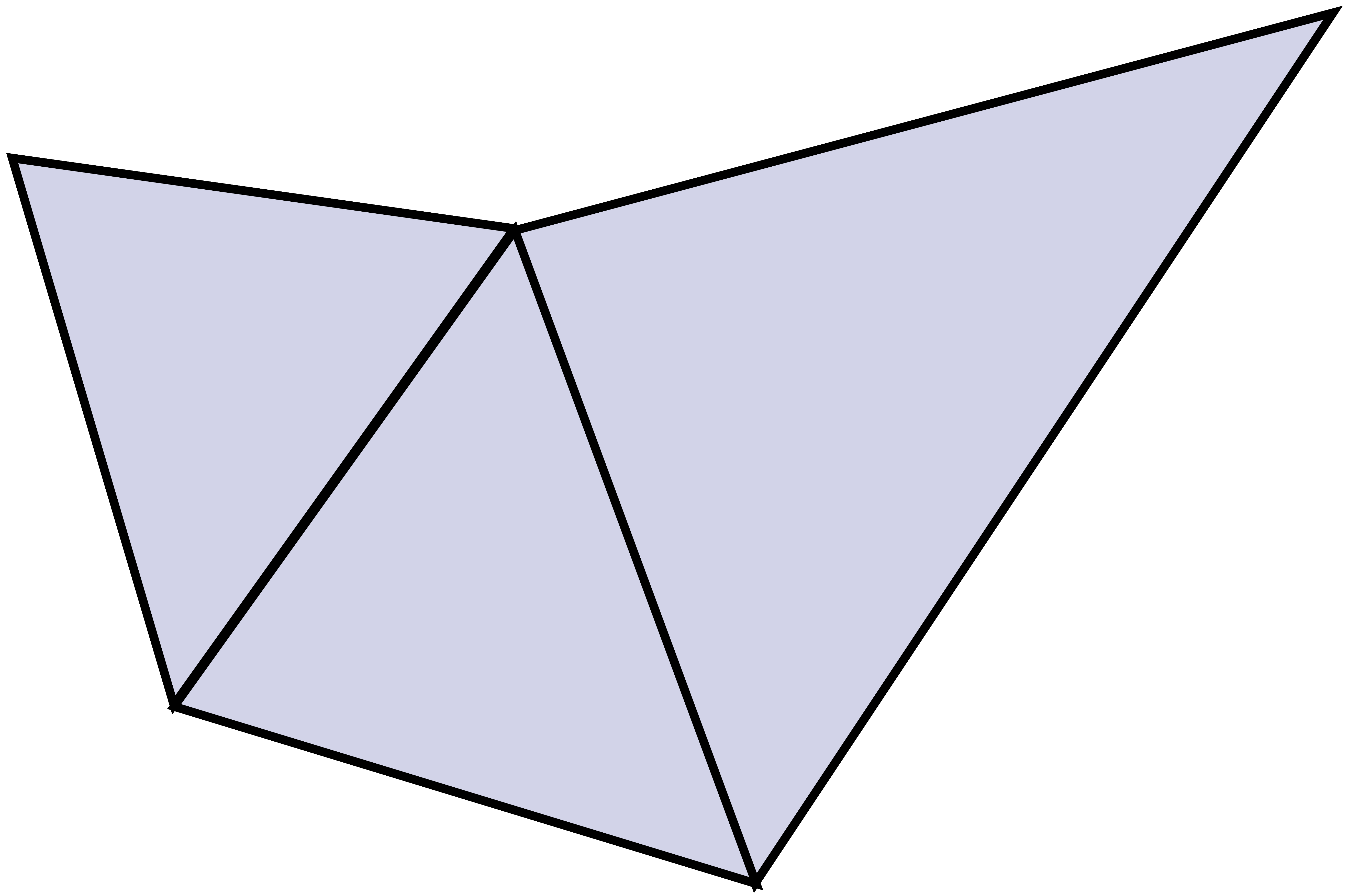
First attempt at a definition for discrete conformal maps:

Definition. A simplicial map between triangulated disks is *conformal* if it preserves interior angles.



Rigidity of Angle Preservation

Problem: one triangle determines the entire map! (Too "rigid")



(…But what if the domain has curvature?)

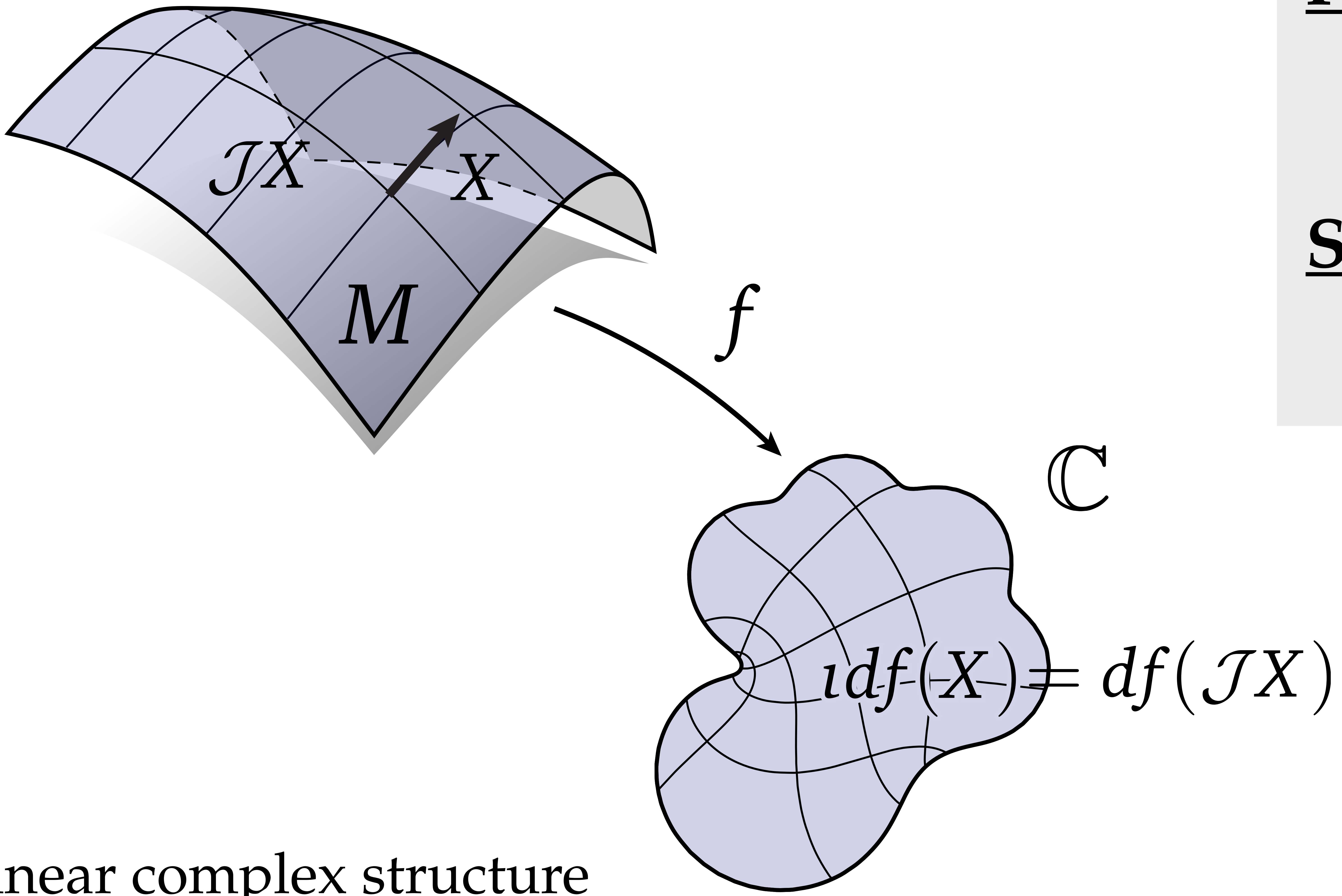
Holomorphic Maps from a Surface to the Plane

<u>Plane to plane:</u>

$$df(\imath X) = \imath df(X)$$

<u>Surface to plane:</u>

$$df(\mathcal{J}X) = \imath df(X)$$

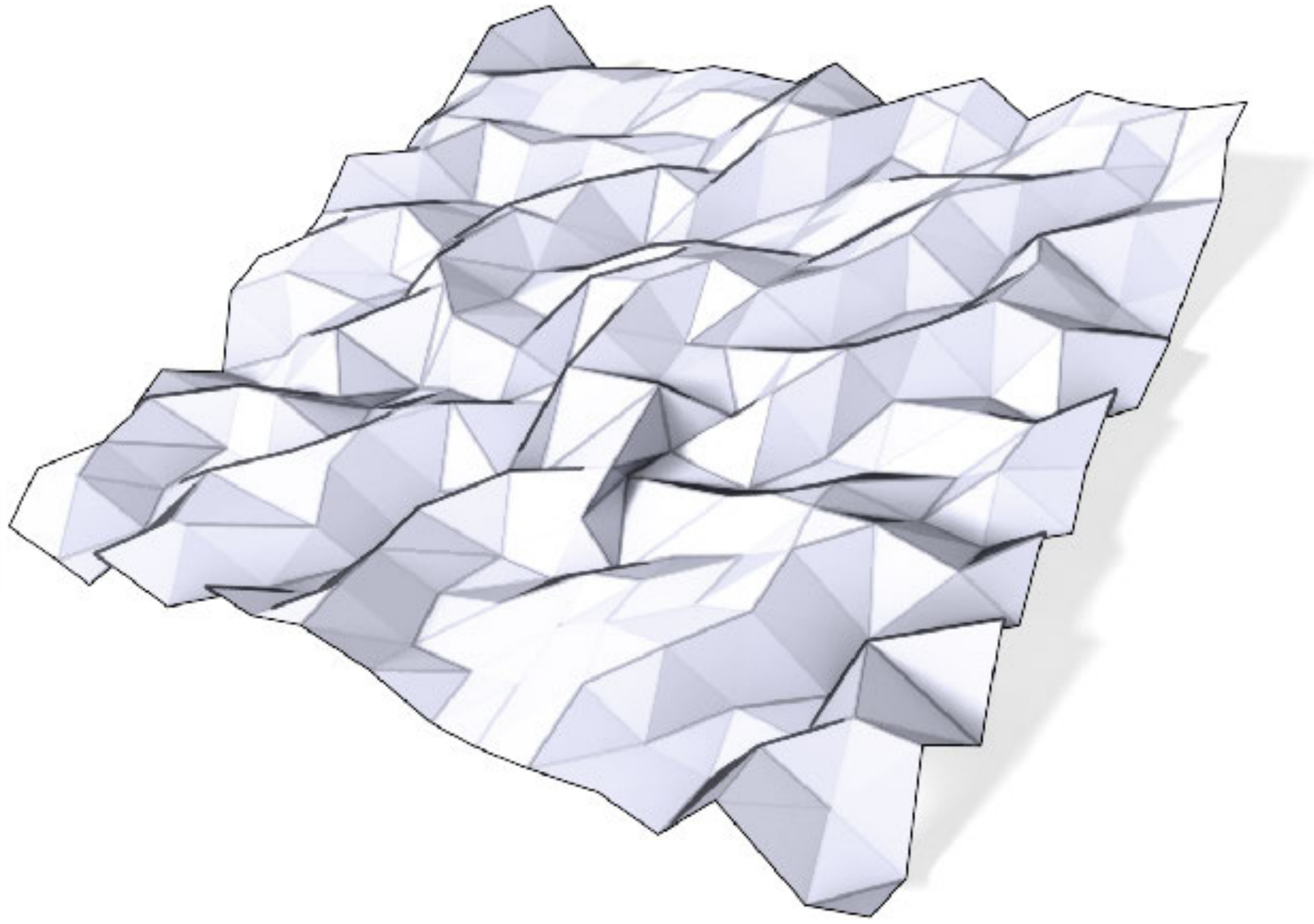


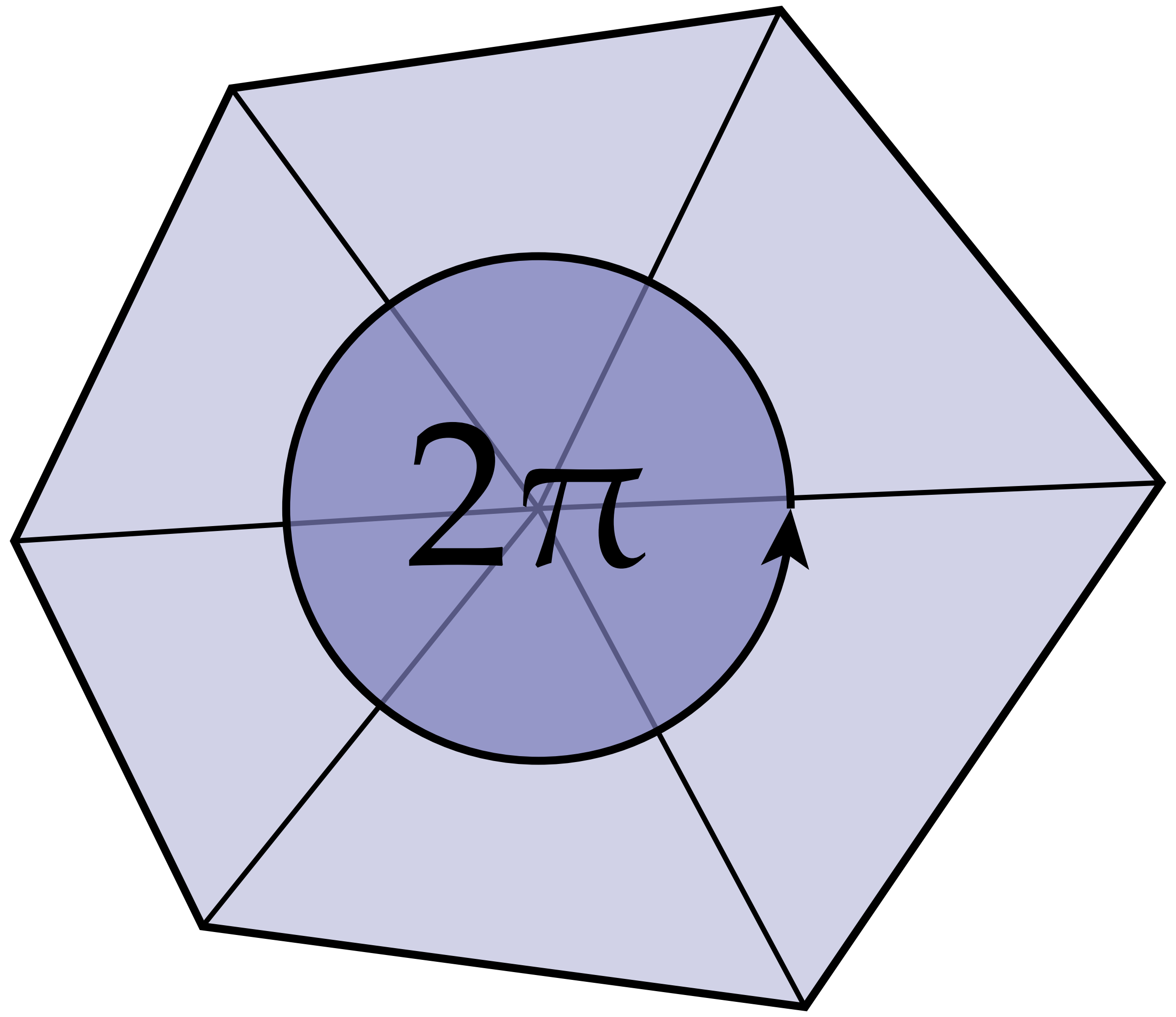
$\mathcal{J}$ — linear complex structure

$\mathcal{J} \circ \mathcal{J} = -\mathrm{id}$

Discrete Angle Preservation (Surface to Plane)

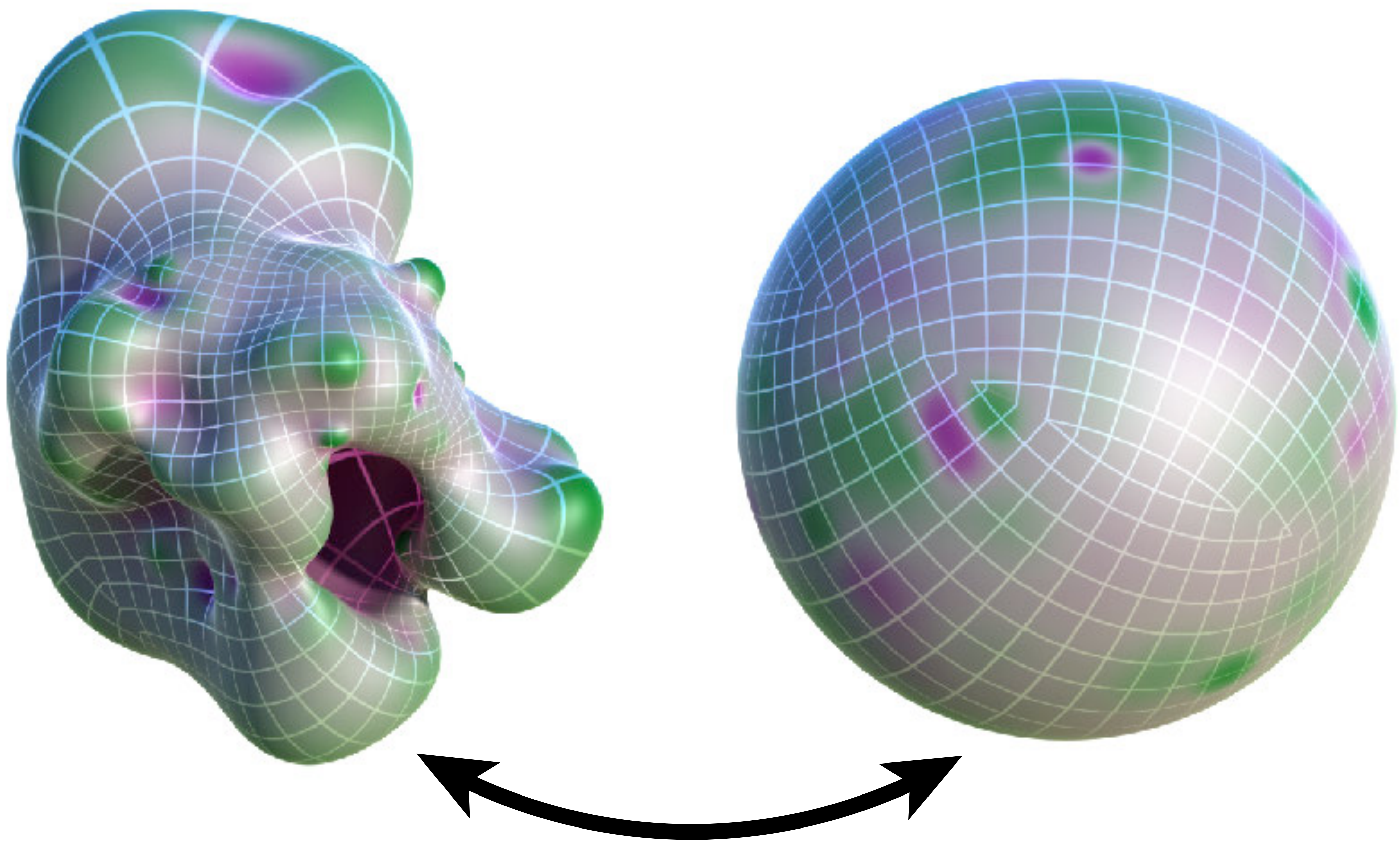
Fact. There is an angle-preserving map from a simplicial disk (K,ℓ) to the plane if and only if the angle sum around every interior vertex is 2π.

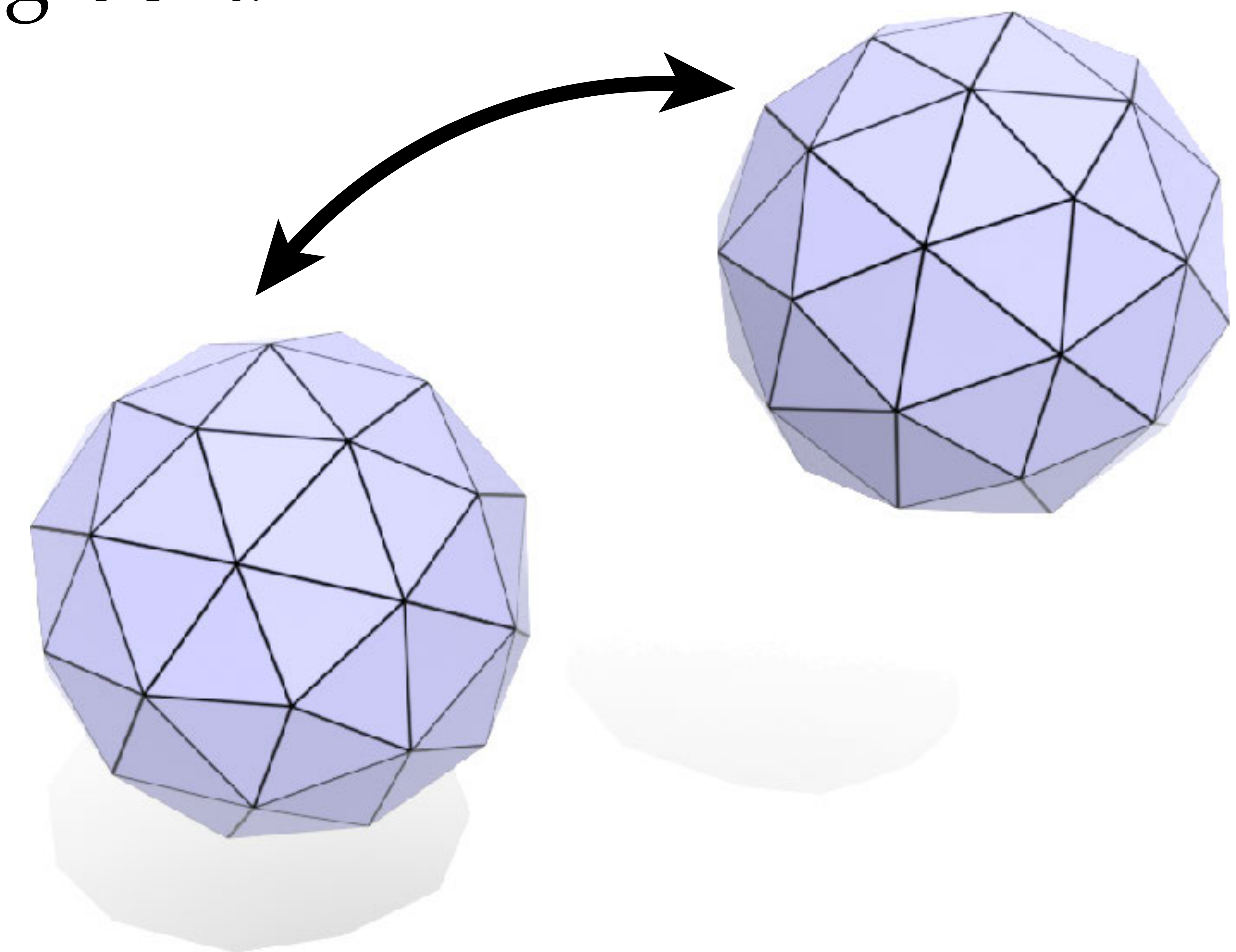




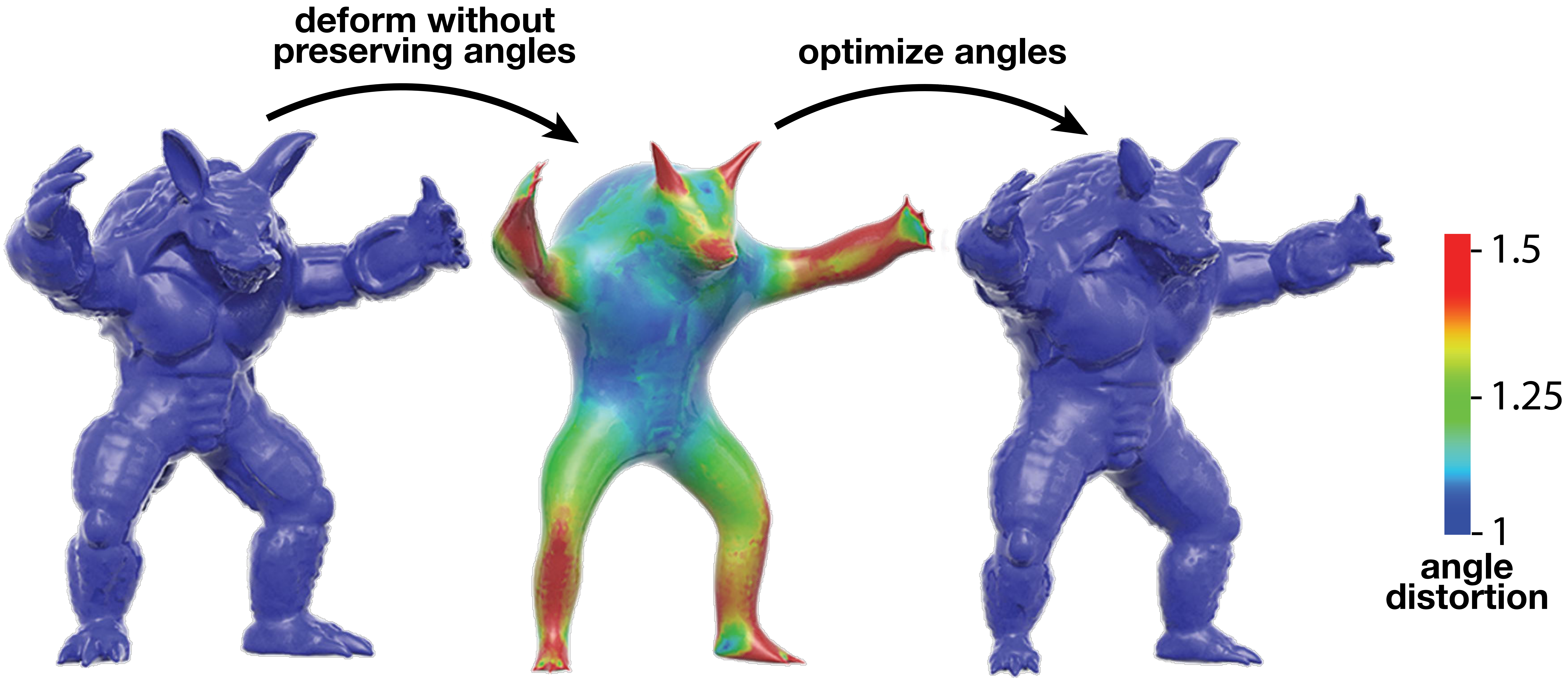
Discrete Angle Preservation (Sphere)

SMOOTH	DISCRETE
Theorem (Uniformization). There exists a conformal map from any topological sphere (M,g) to the unit sphere.	**Theorem** (Cauchy). Convex polytopes with congruent faces are themselves congruent.



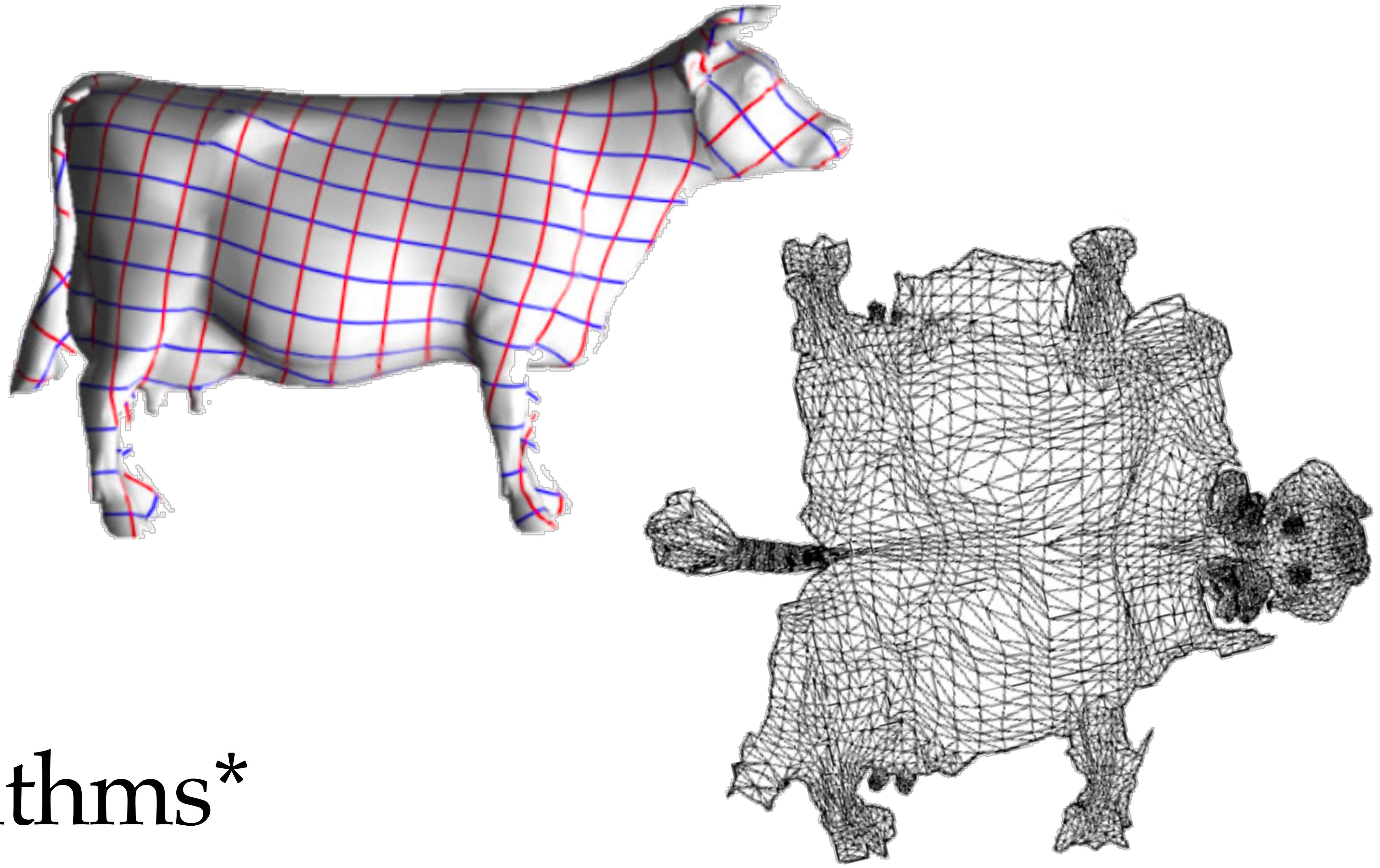


Discrete Angle Preservation (Nonconvex)

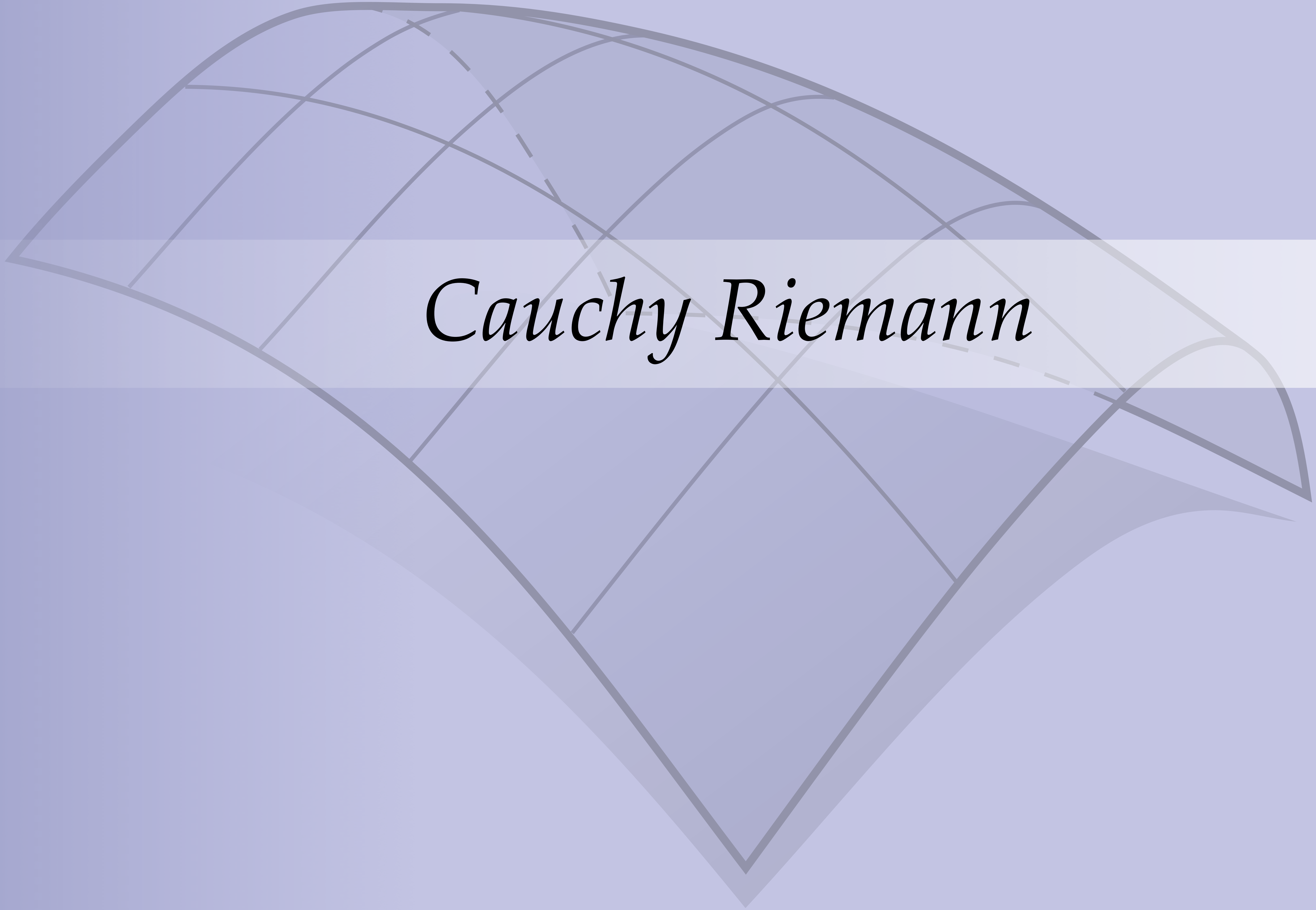


Least Angle Distortion

- **Conclusion:** angle preservation is *too rigid*: in general, an angle-preserving flattening of a given simplicial disk may not exist.
- **Compromise:** define discrete conformal flattening as simplicial map of *least* area distortion
  - now we get at least one such map
  - in general, *only* one map
  - still too rigid!
  - still starting point for practical algorithms*



*SHEFFER, DE STURLER, *"Parameterization of Faceted Surfaces for Meshing using Angle Based Flattening"* (2001)



Cauchy Riemann

Discretization via Cauchy Riemann

- Angle preservation was too rigid
- Why not start from the traditional *Cauchy-Riemann* equation?
- Perhaps most traditional way to characterize holomorphic maps:

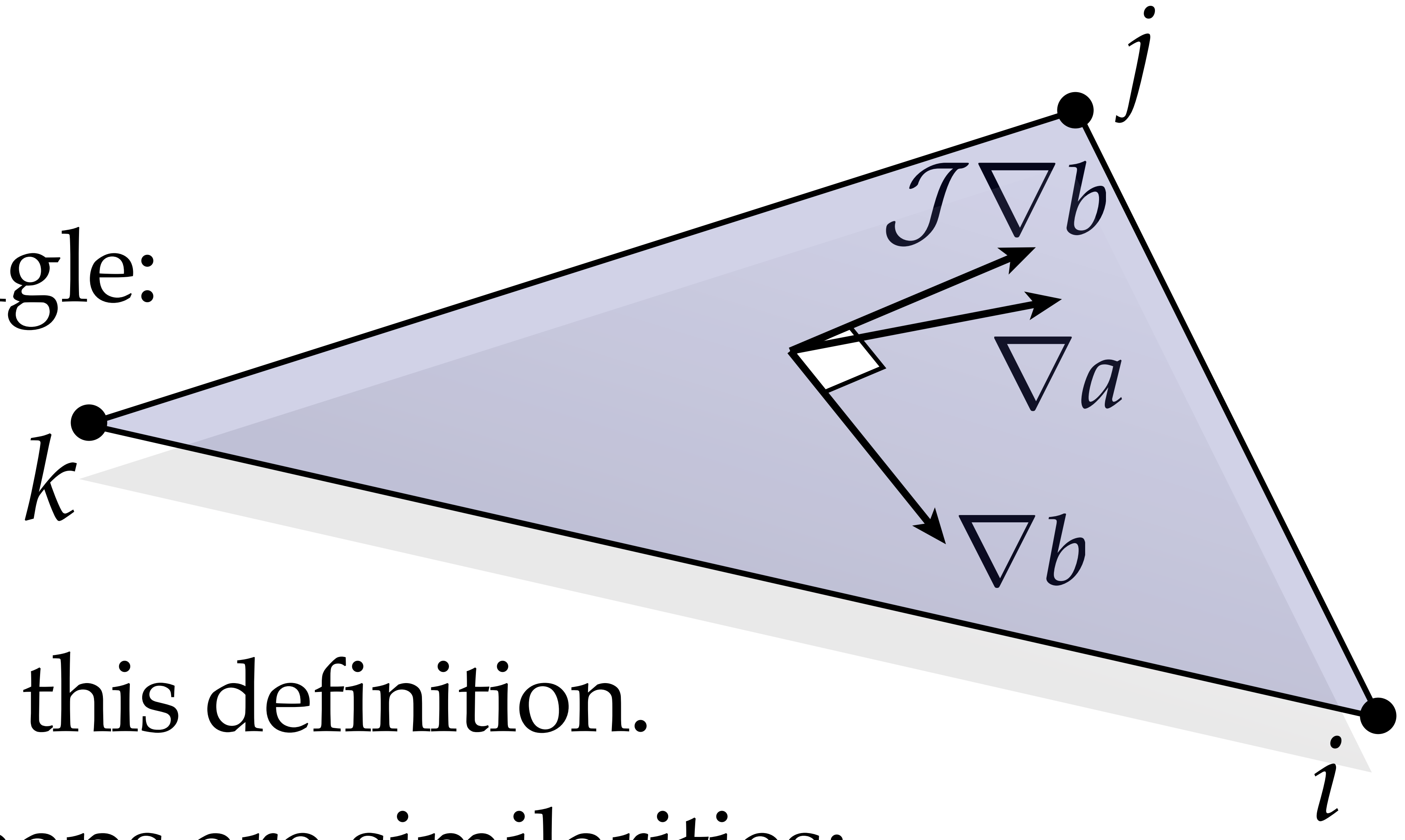
> A map $f : \mathbb{C} \supset U \to \mathbb{C}$ given by $(x, y) \mapsto a(x, y) + \imath b(x, y)$ is **holomorphic** if at each point of U the coordinate functions $a, b : \mathbb{C} \to \mathbb{R}$ satisfy
>
> $$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} \qquad \text{and} \qquad \frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}.$$

- **Idea:** simply define discrete conformal maps as piecewise linear maps that are holomorphic when restricted to each simplex.

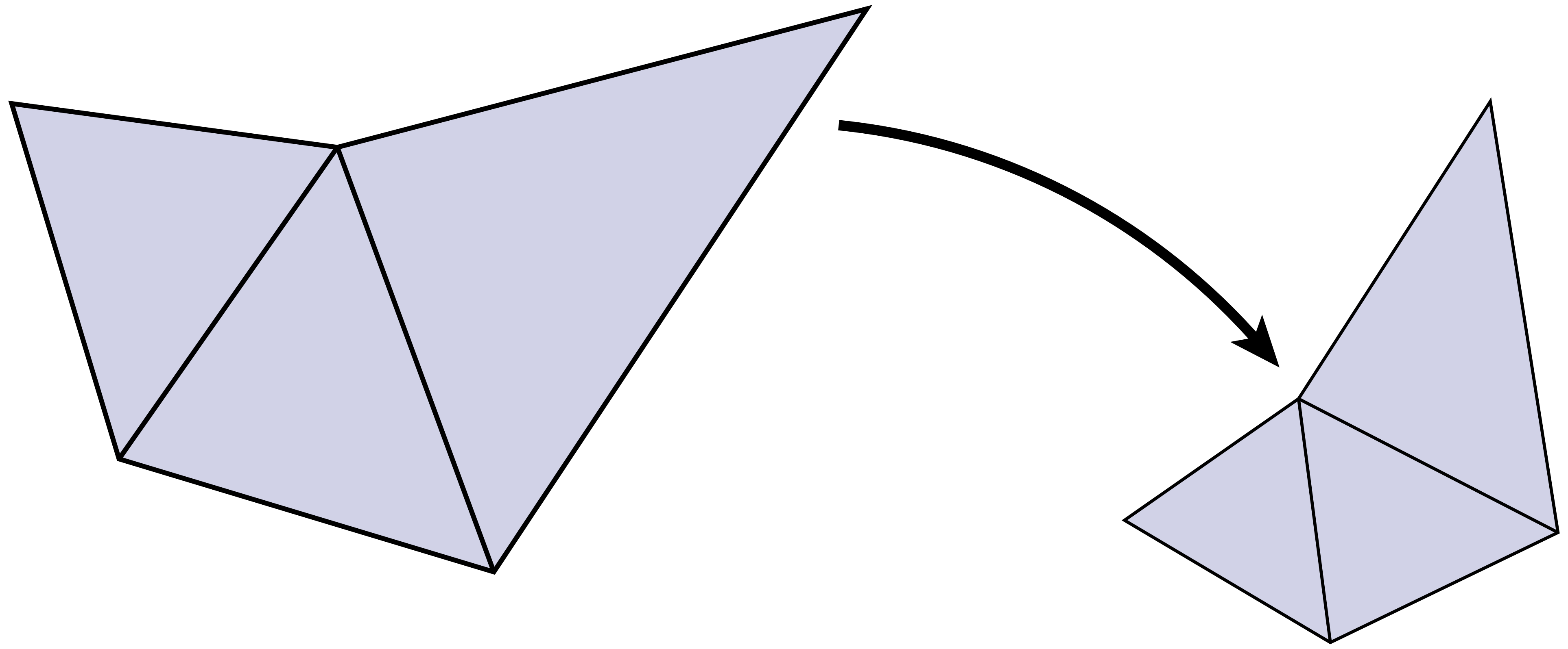
Discretization via Cauchy Riemann

- Put real coordinates a_i, b_i at each vertex i
- Linearly interpolate over triangles
- Require Cauchy-Riemann to be satisfied in each triangle:

$$(\nabla a)_{ijk} = (\mathcal{J}\nabla b)_{ijk} \qquad \forall ijk \in F$$



- For most surfaces, will not find *any* map that satisfies this definition.
- Should not be a surprise! Only affine holomorphic maps are similarities:

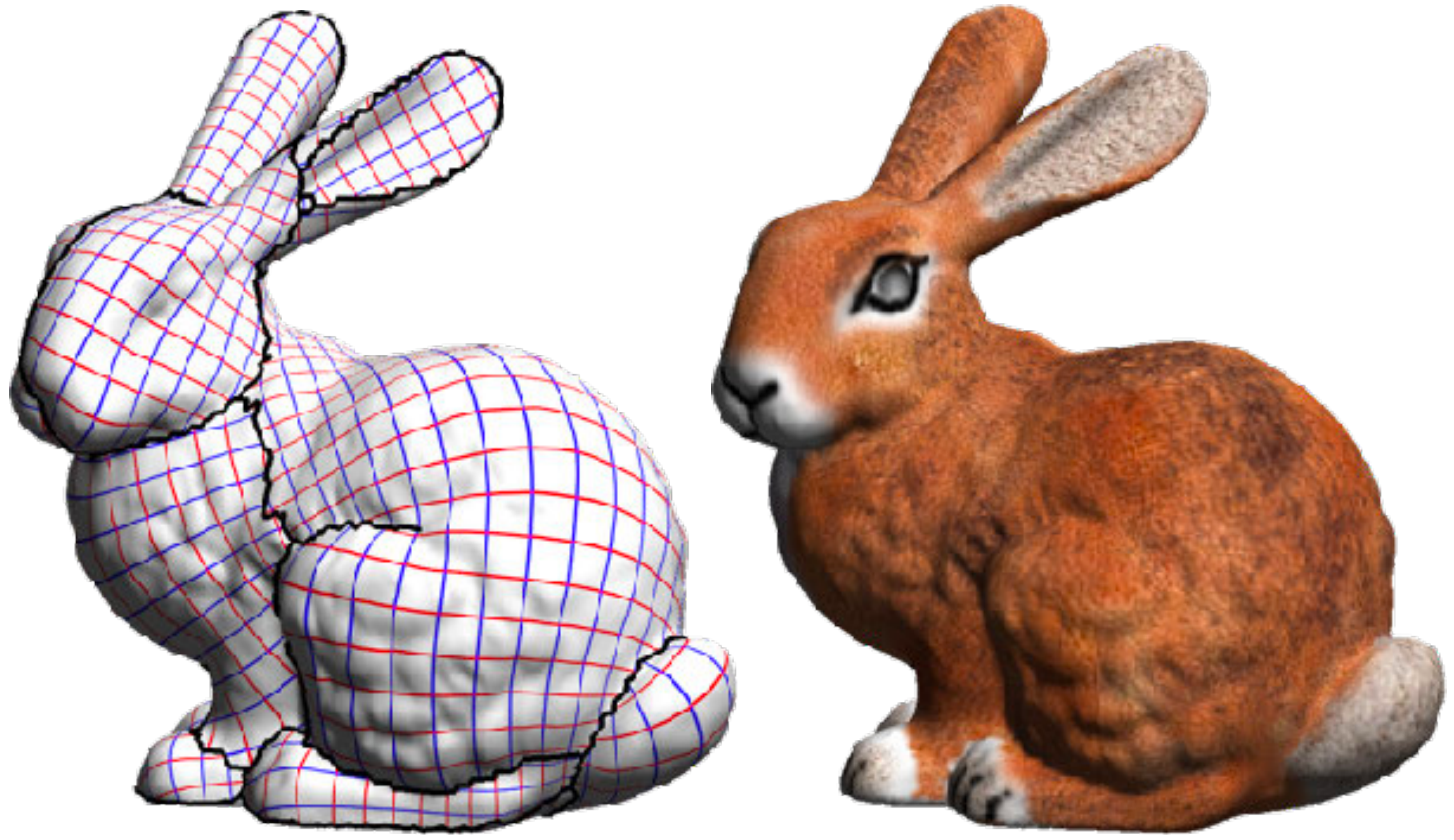


(Same as angle preservation.)

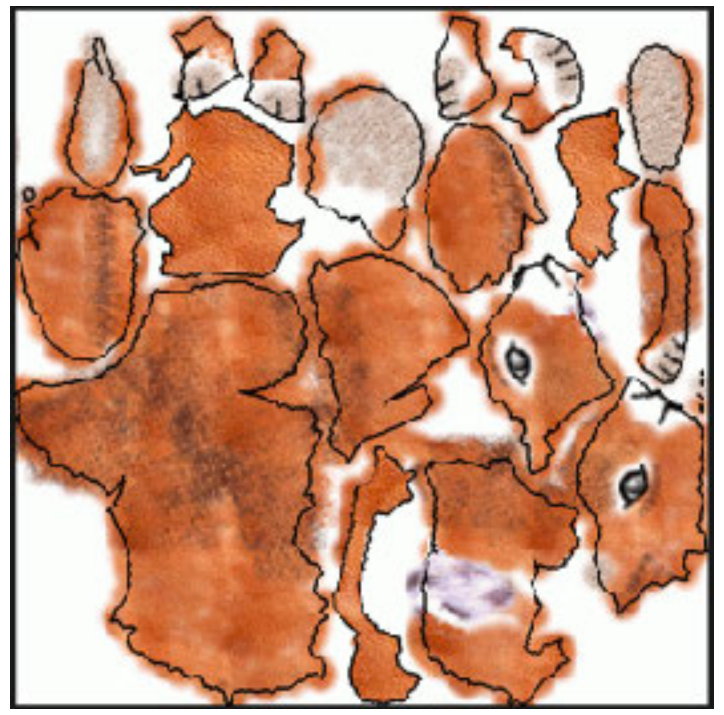
Least Square Conformal Maps

- **Compromise.** As with angle preservation, can still seek "most holomorphic map."
- Sum residual of Cauchy Riemann over triangles to get *least squares conformal energy:*

$$E_{\mathrm{LSCM}}(a,b) := \sum_{ijk\in F} \mathcal{A}_{ijk}\left|(\nabla a)_{ijk} - (\mathcal{J}\nabla b)_{ijk}\right|^2$$



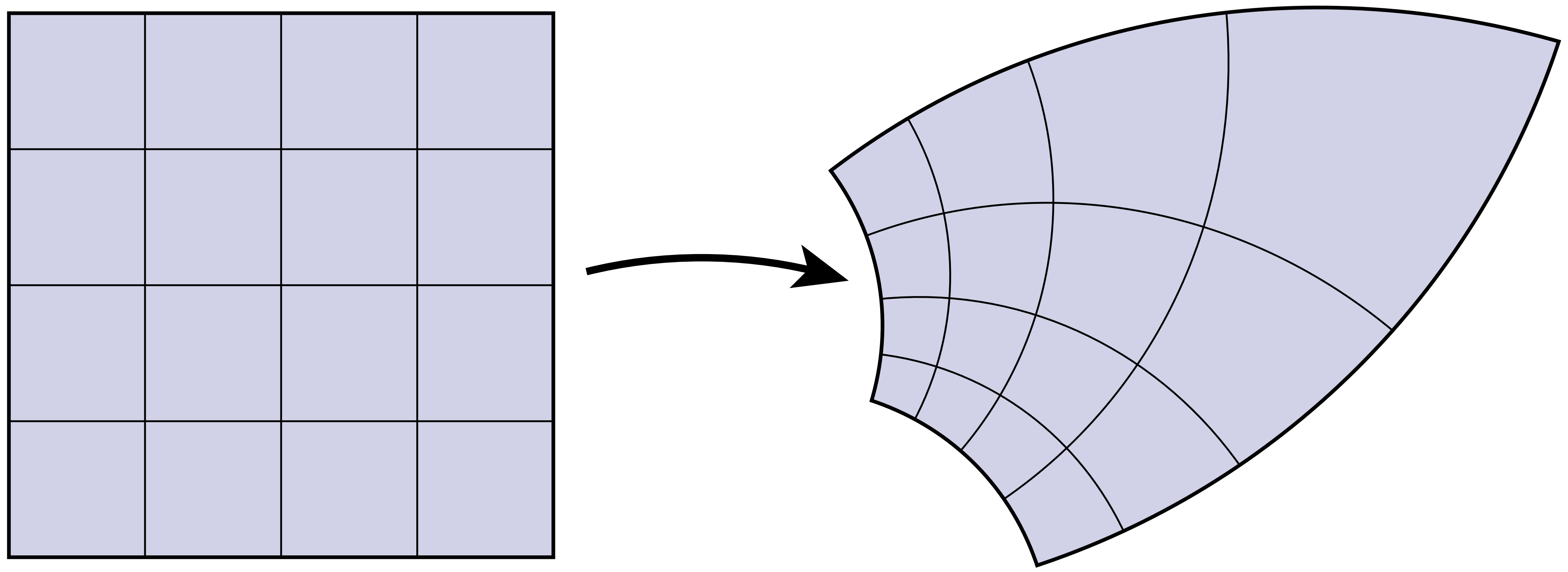
- Fix rigid motion (rotation + scaling in plane) by specifying map at two vertices
- Resulting energy is convex and *strictly* quadratic
- Hence, **still too rigid**: always get a *unique* minimizer
- Yet still most commonly-used technique used in practice*
- Why? Requires only a single linear solve (*fast*)

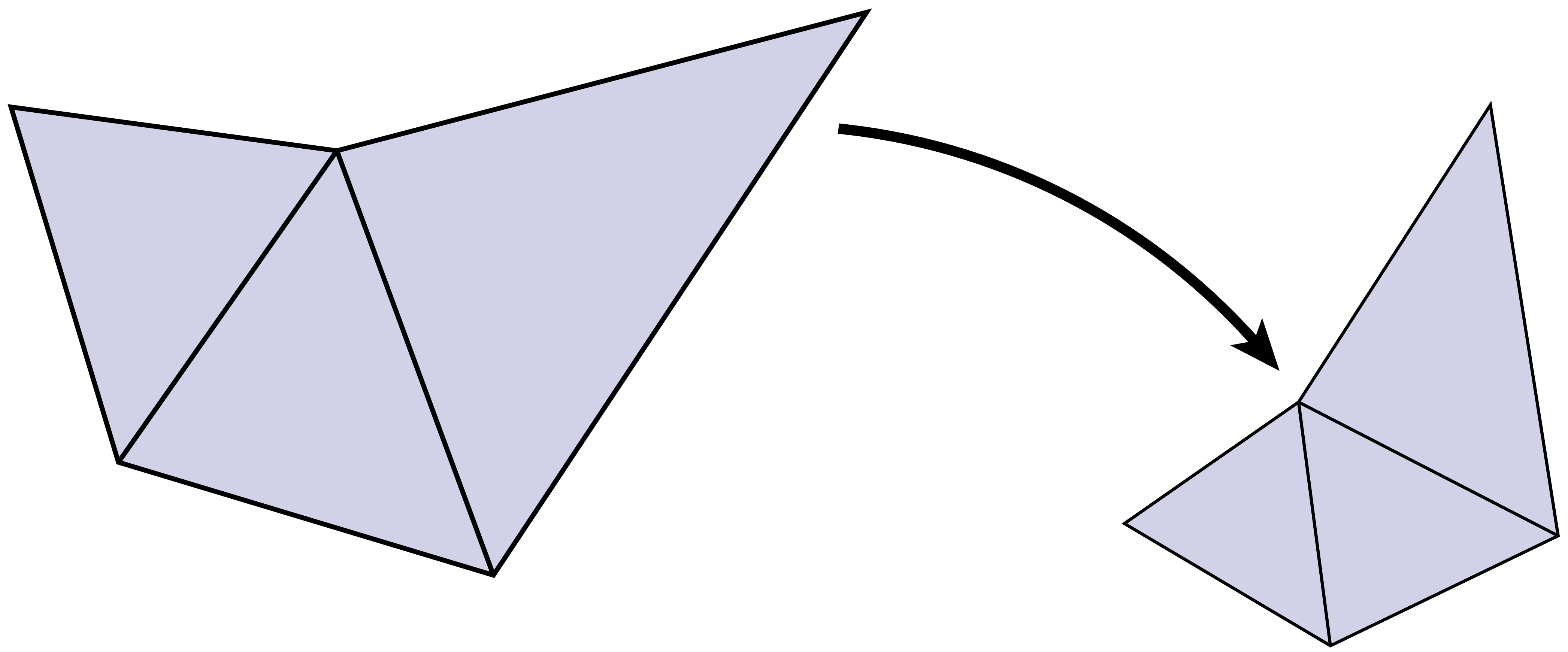


*LÉVY, PETITJEAN, RAY, MAILLOT, *"Least Squares Conformal Maps for Automatic Texture Atlas Generation"* (2011)

Beyond the Basics

- So far, our two most basic approaches have failed:
  - **angle preservation** ⇒ *too rigid*
  - **piecewise holomorphic** ⇒ *too rigid*
- At this point, starts to become clear that one cannot be naïve about discretization
- Really need to think more deeply about relationship between smooth & discrete geometry

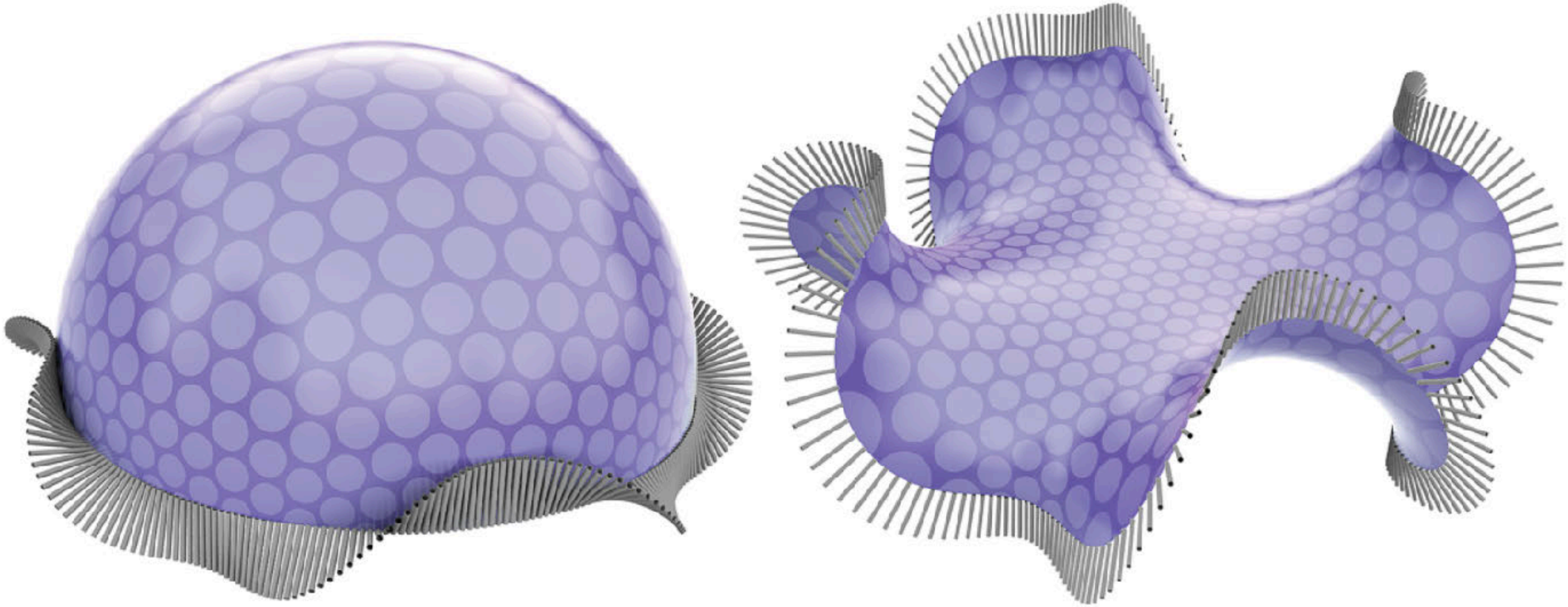




Circle Preservation

Circle Preservation

- **Smooth**: conformal maps preserve infinitesimal circles
- **Discrete**: try to preserve circles associated with mesh elements

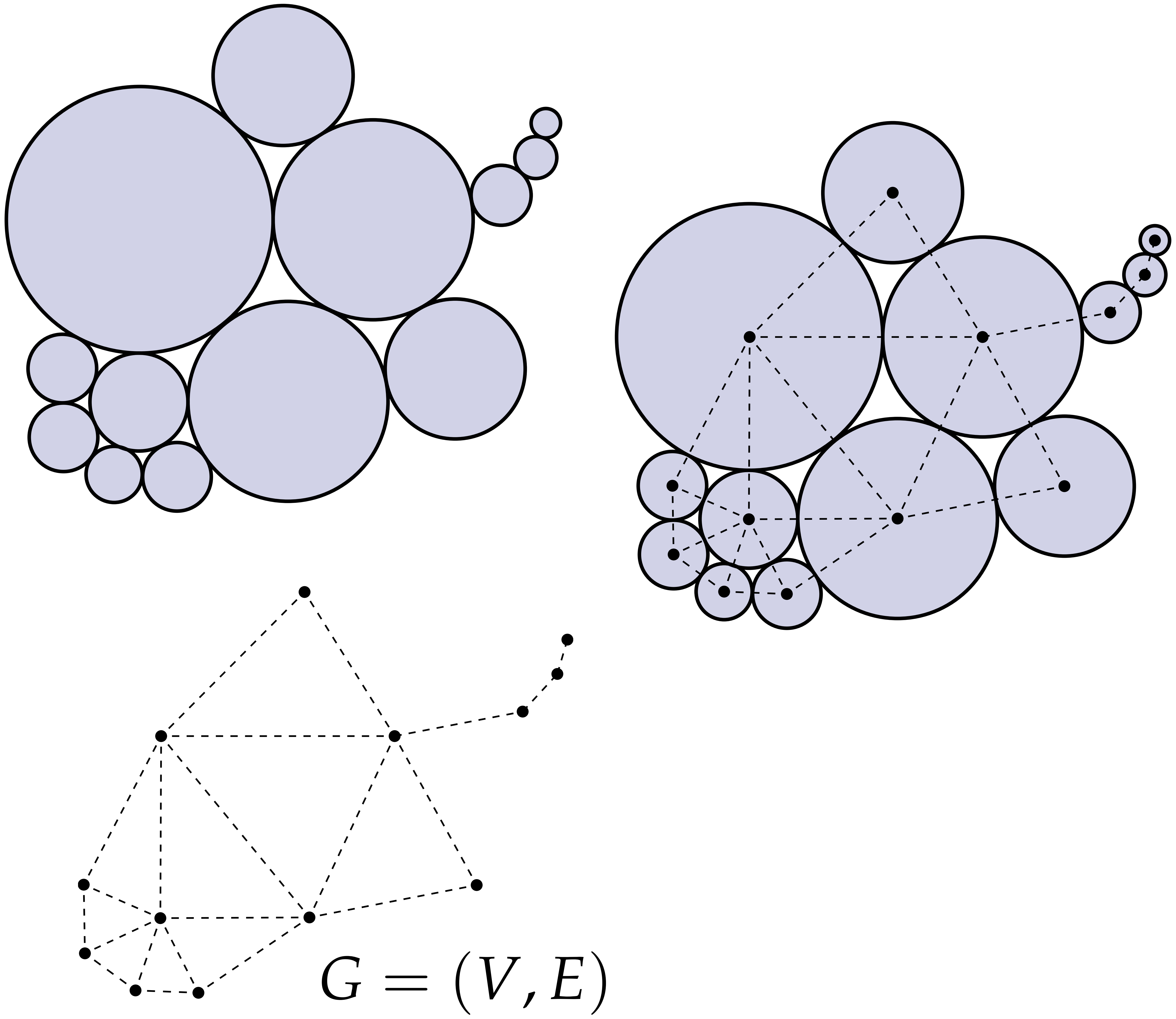


- First perspective that "really starts to work" (& important early example in DDG)
- Still won't get us all the way there…

Circle Packing

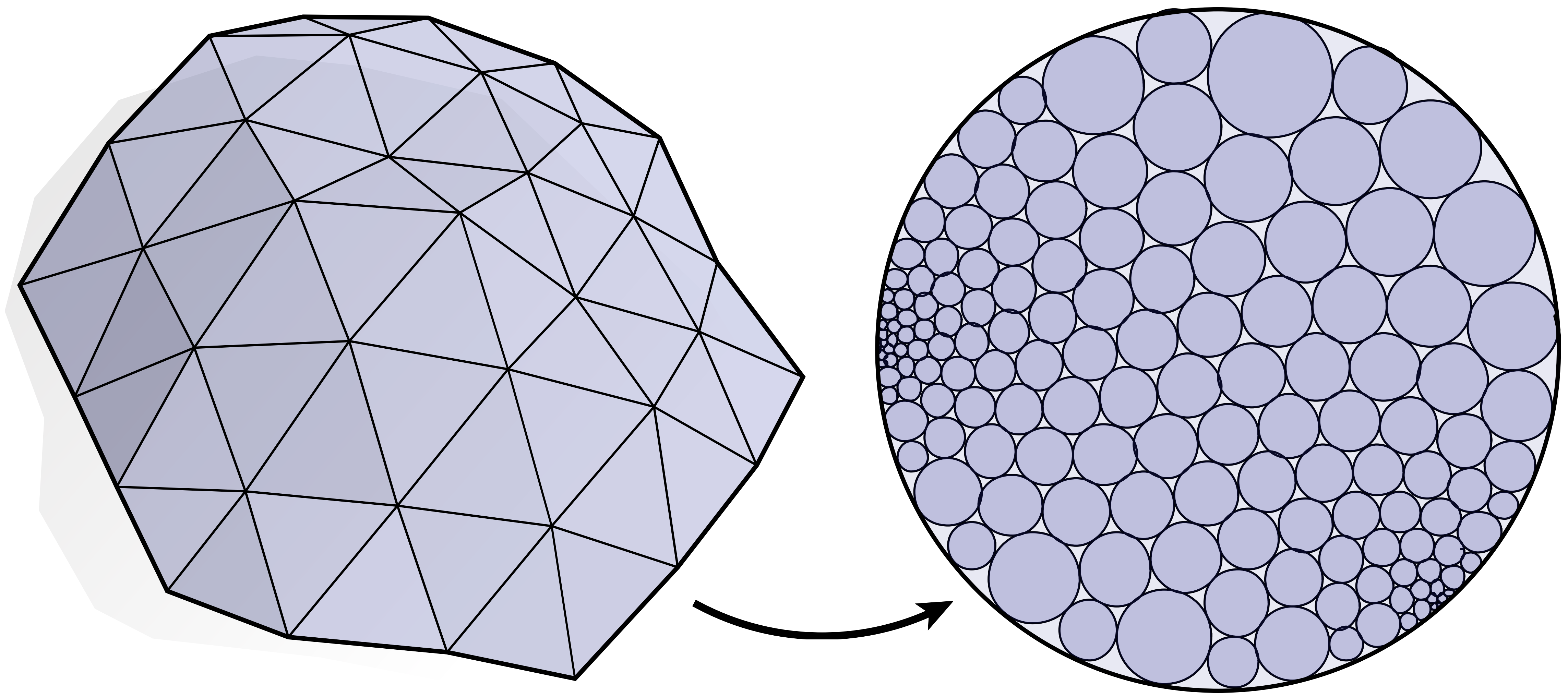
- A **circle packing** is a collection of closed circular disks in the plane (or other surface) with mutually disjoint interiors.

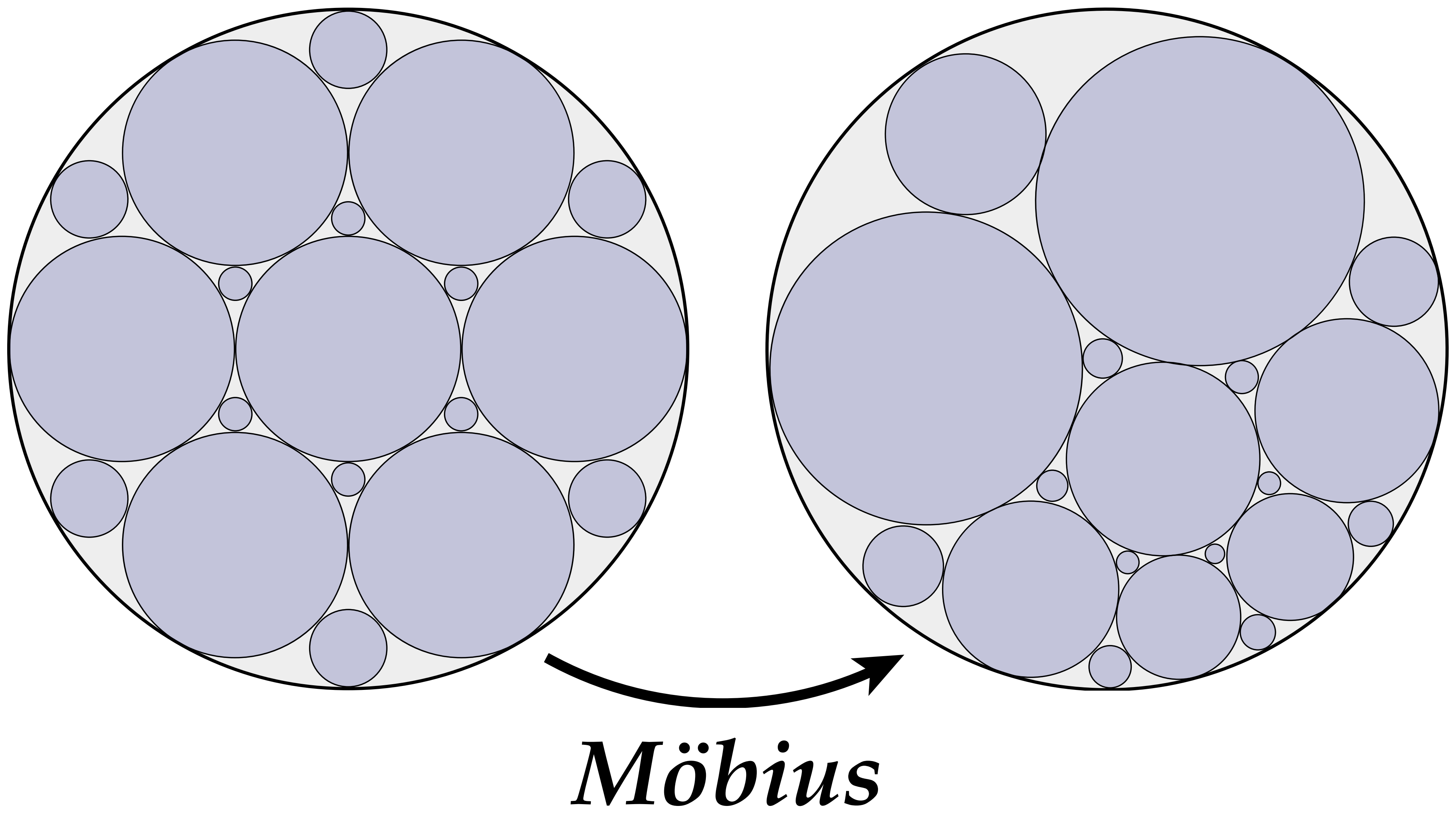
- To any such collection one can associate a graph $G = (V, E)$ where
  - each vertex corresponds to a disk
  - two vertices are connected by an edge if and only if their associated disks are tangent



Circle Packing Theorem

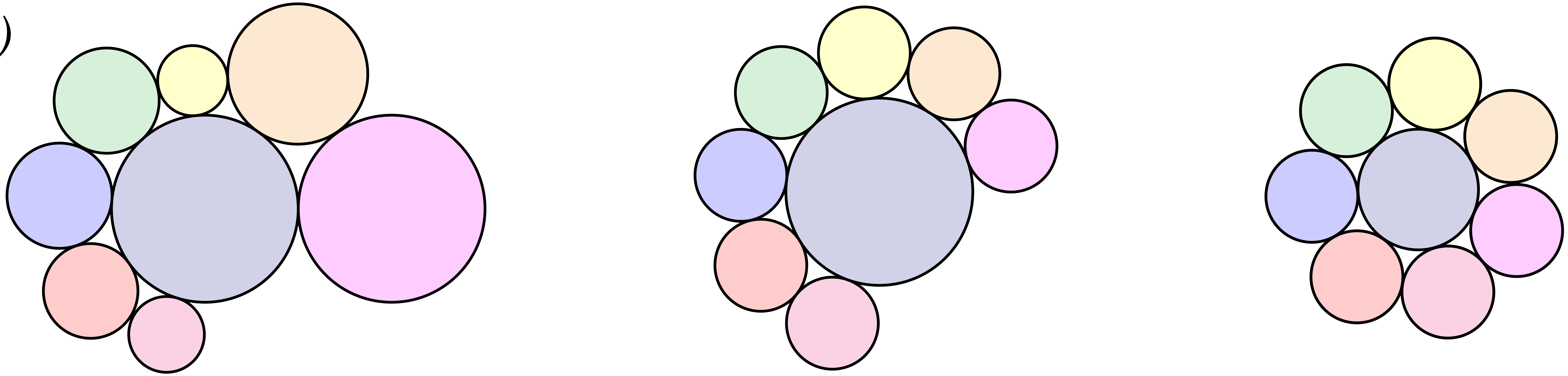
- Which graphs admit a circle packing?
- **Theorem** (Circle Packing). Every planar graph $G = (V,E)$ can be realized as a circle packing in the plane.
- **Theorem** (Koebe 1936). If G is a finite maximal planar graph, then a circle packing of G—*including the outer face*—is unique up to Möbius transformations and reflections.





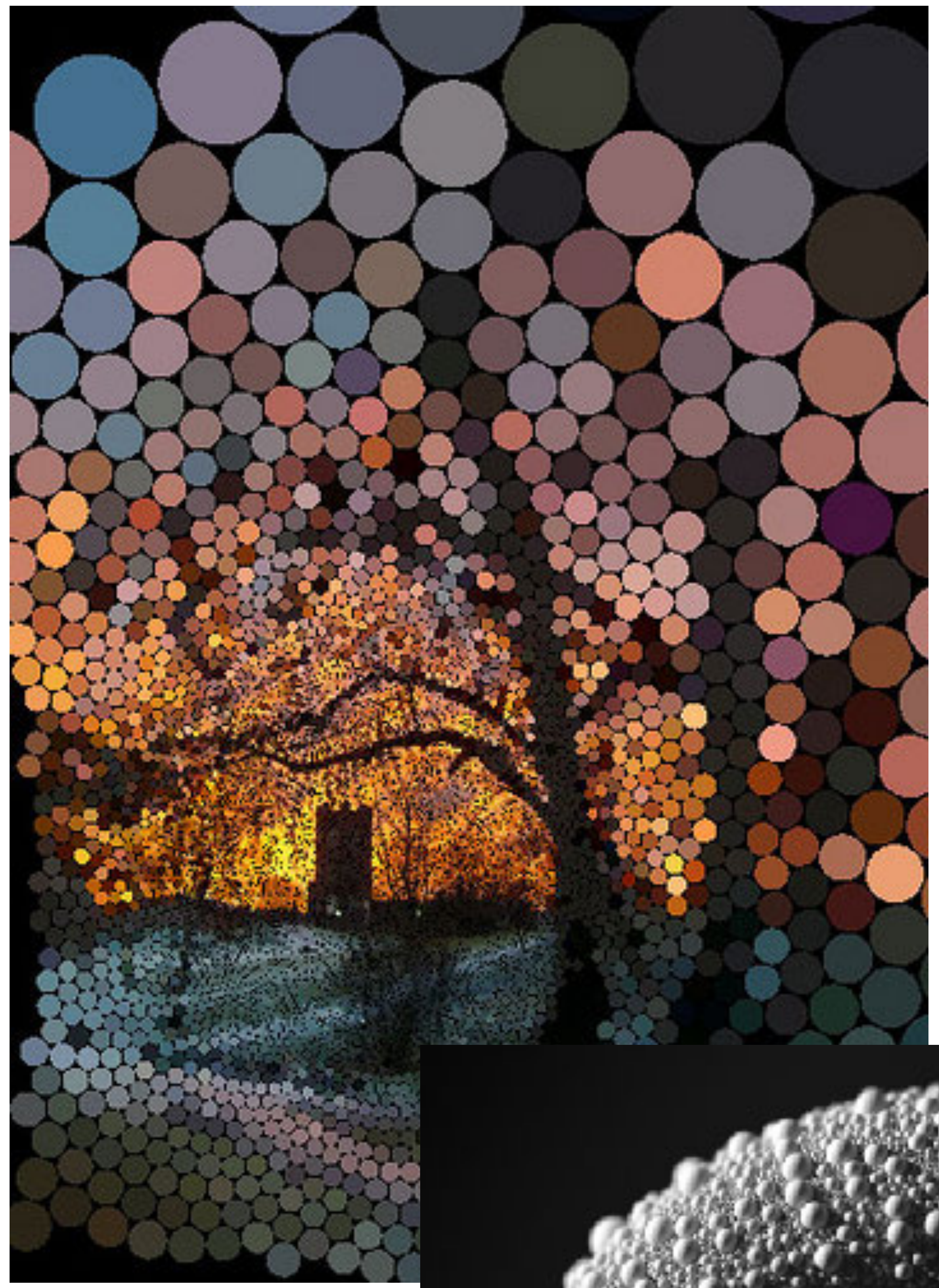
Circle Packing — Algorithm

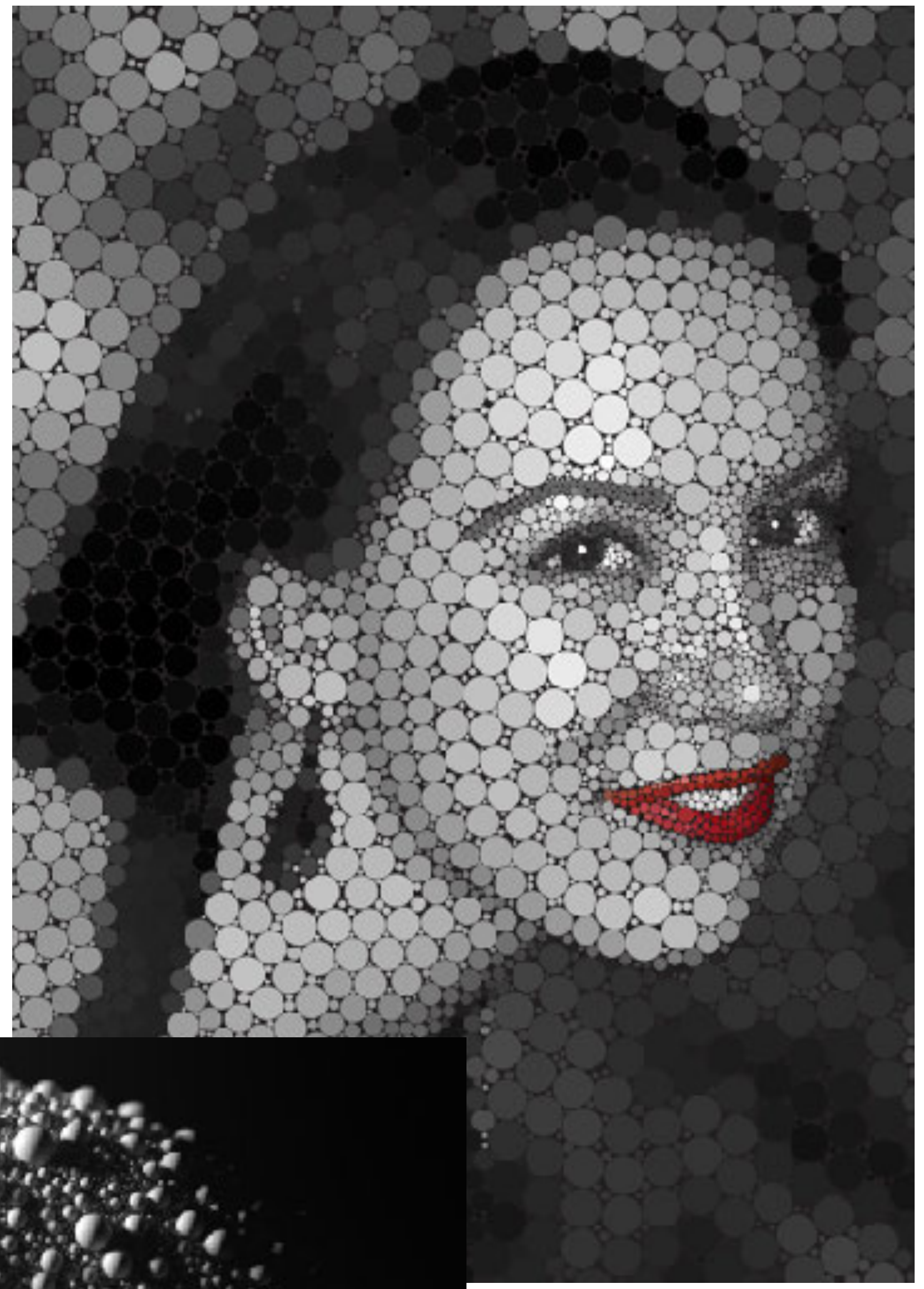
- Nonlinear problem, but simple iterative algorithm*
- For each vertex *i*:
  - Let θ be total angle currently covered by *k* neighbors
  - Let *r* be radius such that *k* neighbors of radius *r* also cover θ
  - Set new radius of *i* such that *k* neighbors of radius *r* cover 2π
- *(Repeat)*

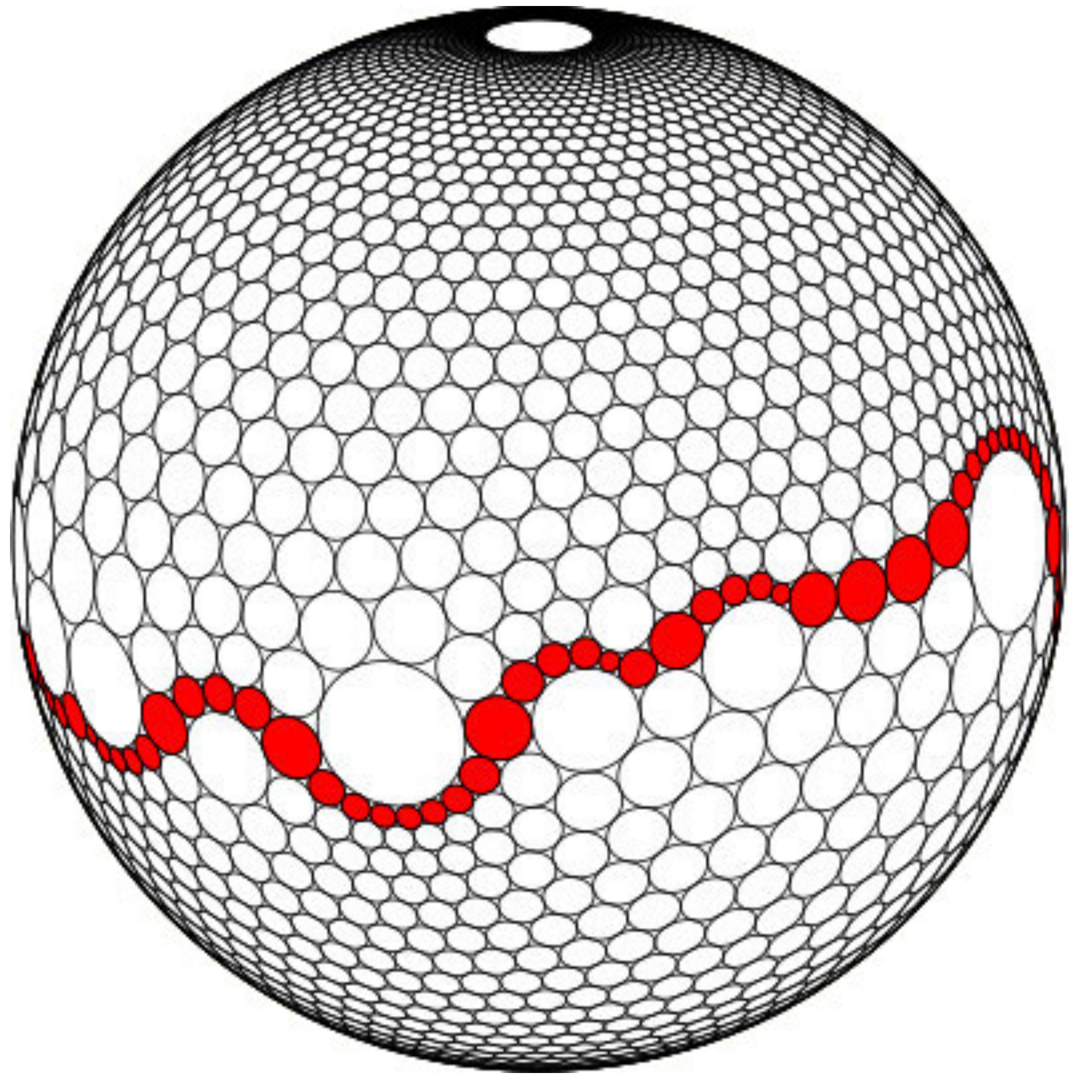


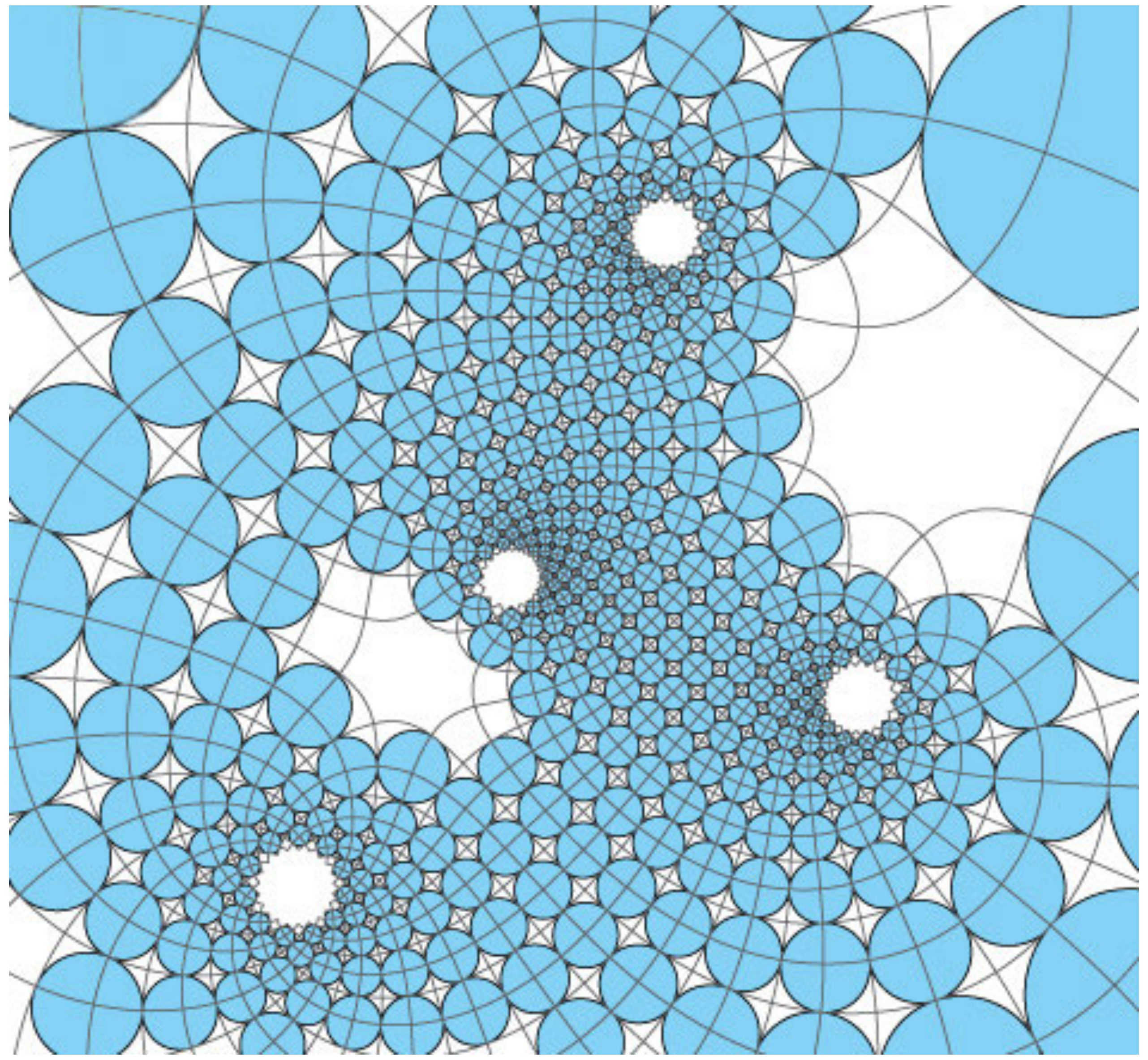
*Collins, Stephenson, *"A Circle Packing Algorithm"* (2003)

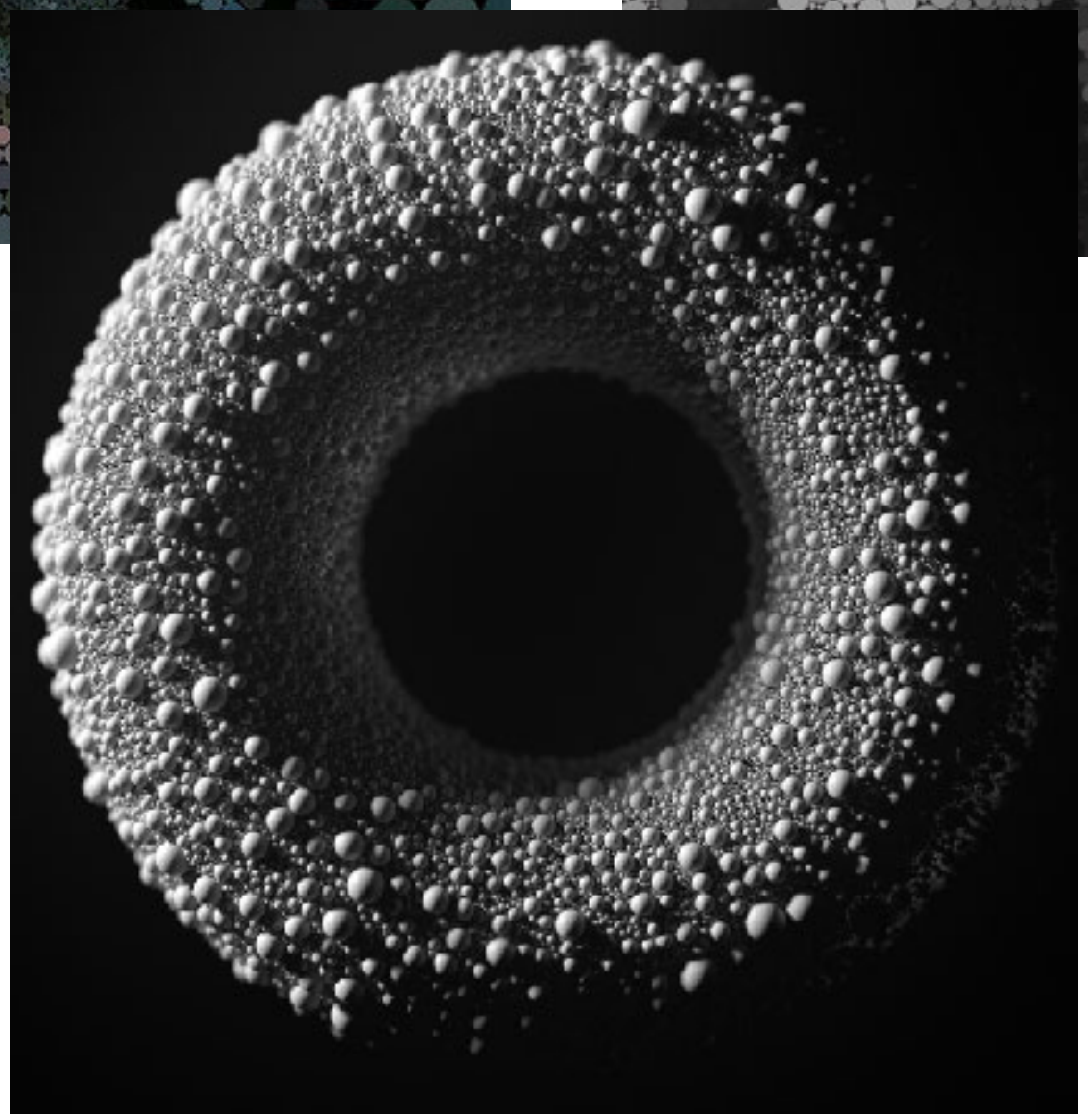
Circle Packing — Gallery

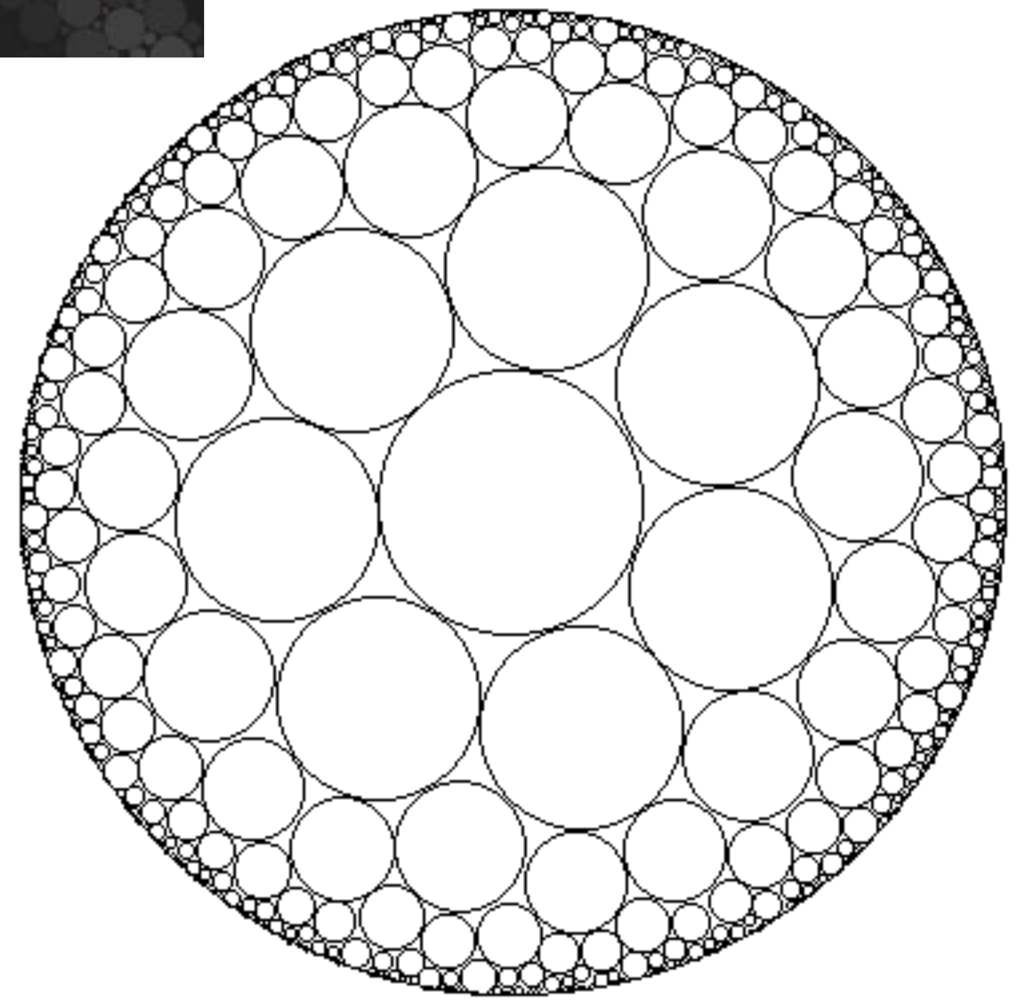


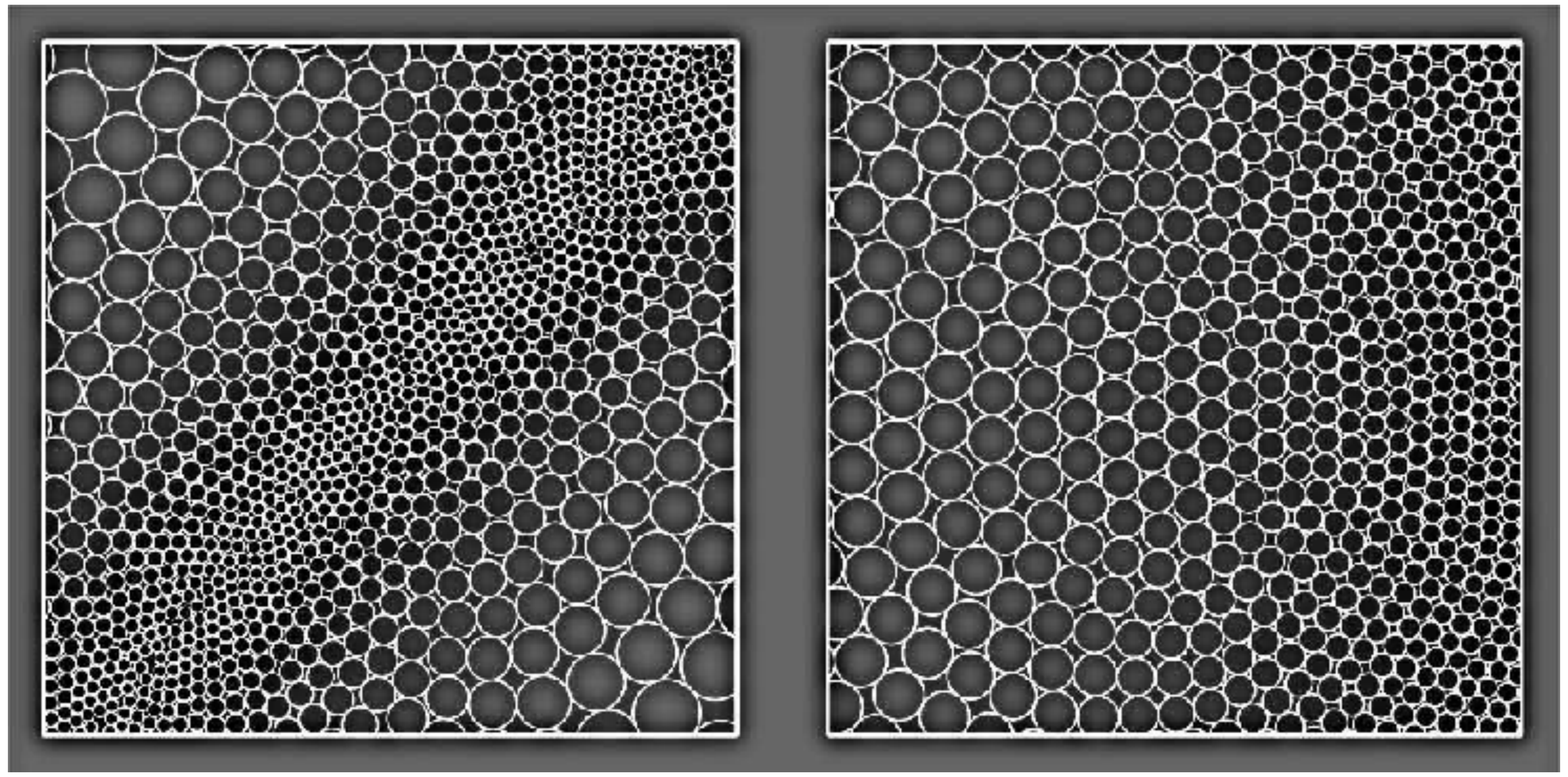






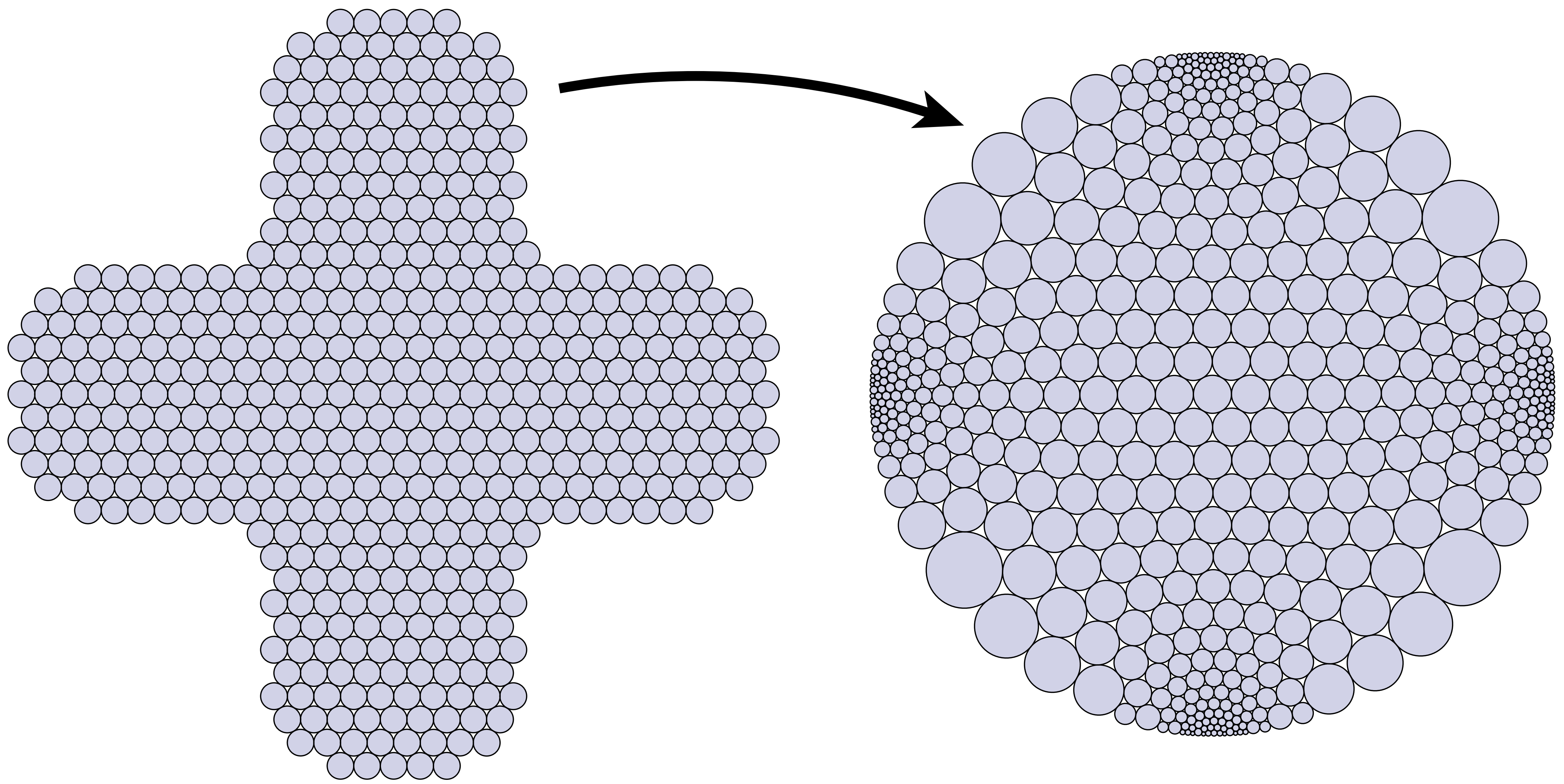


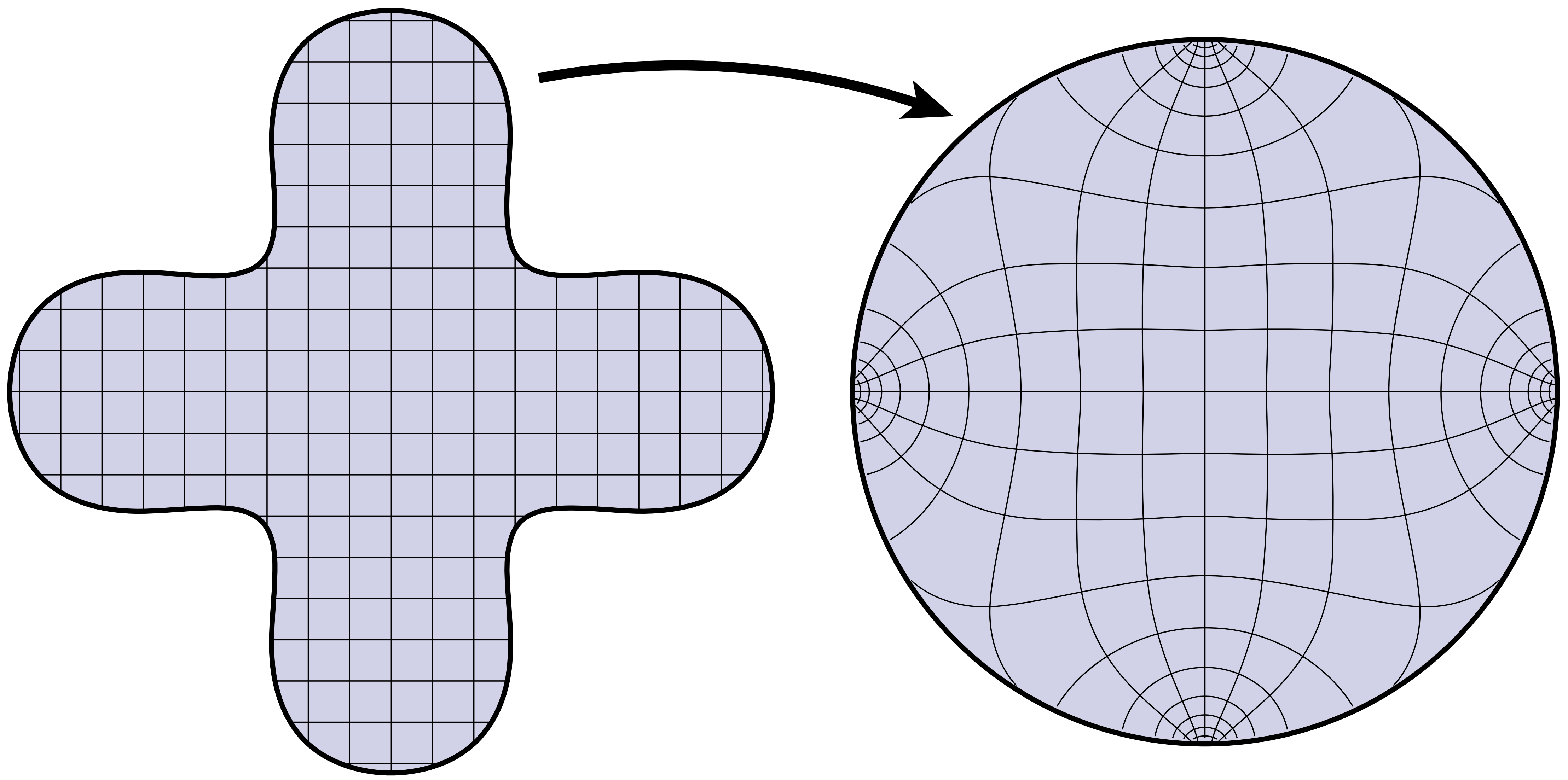




Thurston Circle Packing Conjecture

- W. Thurston (1985): circle packing of a regular hexagonal tiling of a region in the plane appears to approximate a smooth conformal map (Riemann map):

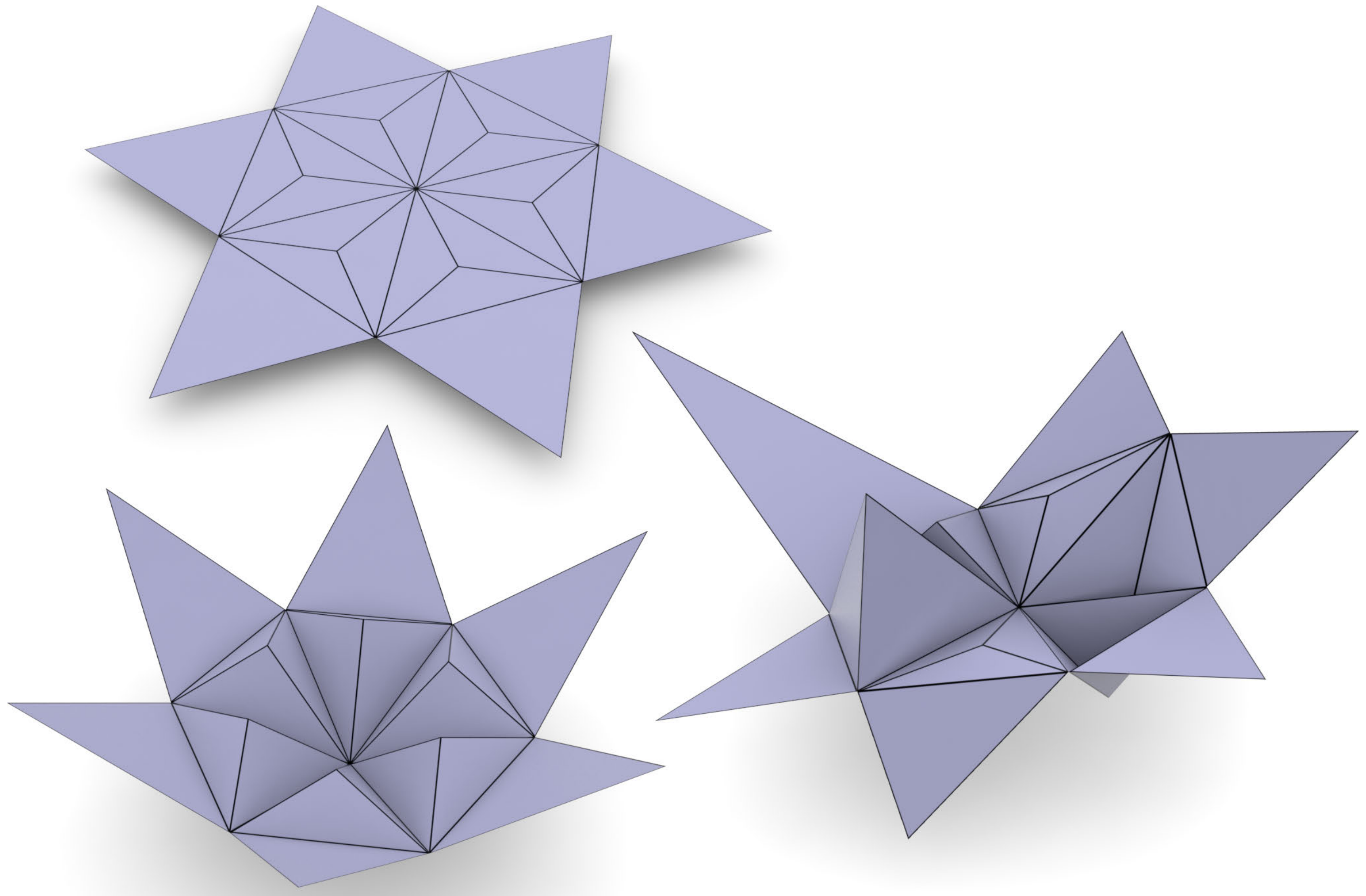


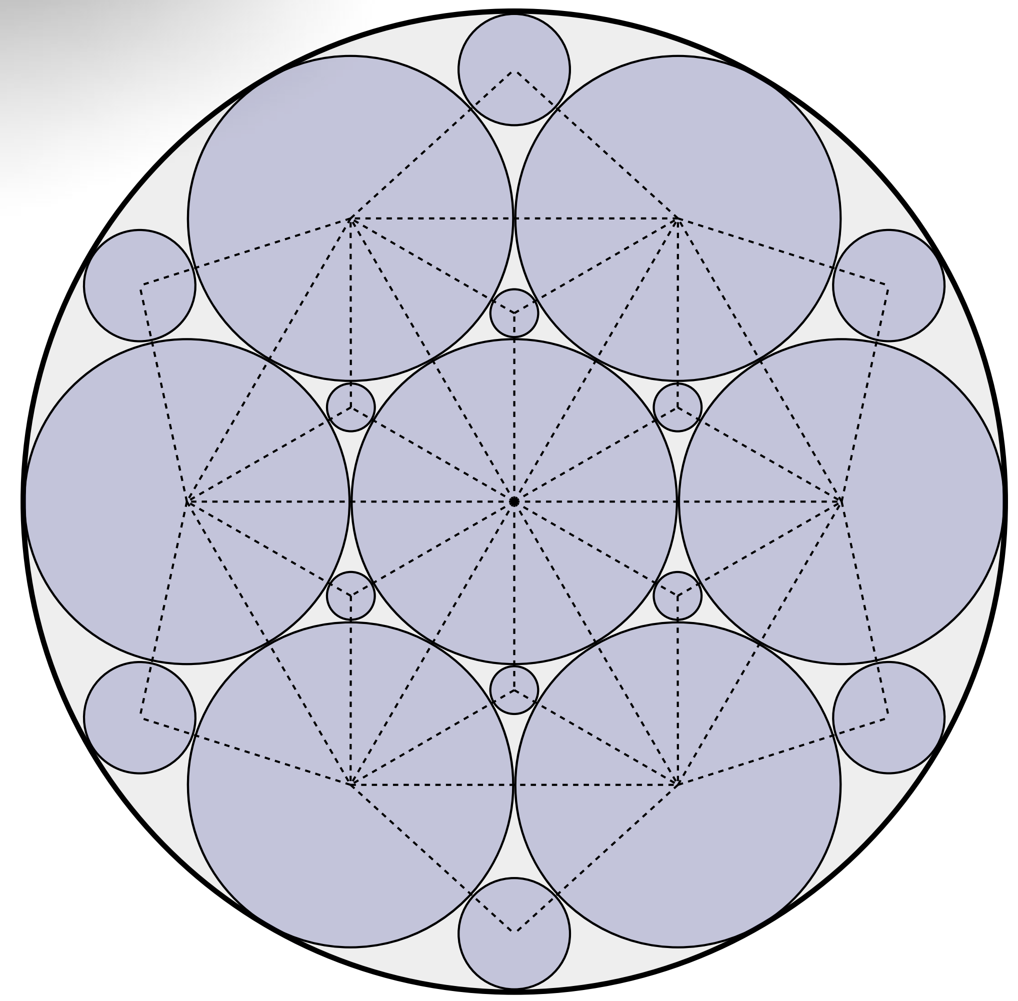


- Rodin & Sullivan (1987): for sufficiently small ε, any point z in domain is covered by a circle c; map it to the center of the corresponding circle c' in the target packing. This *approximation mapping* converges to a Riemann mapping as $\varepsilon \longrightarrow 0$.

Discretization via Circle Packing

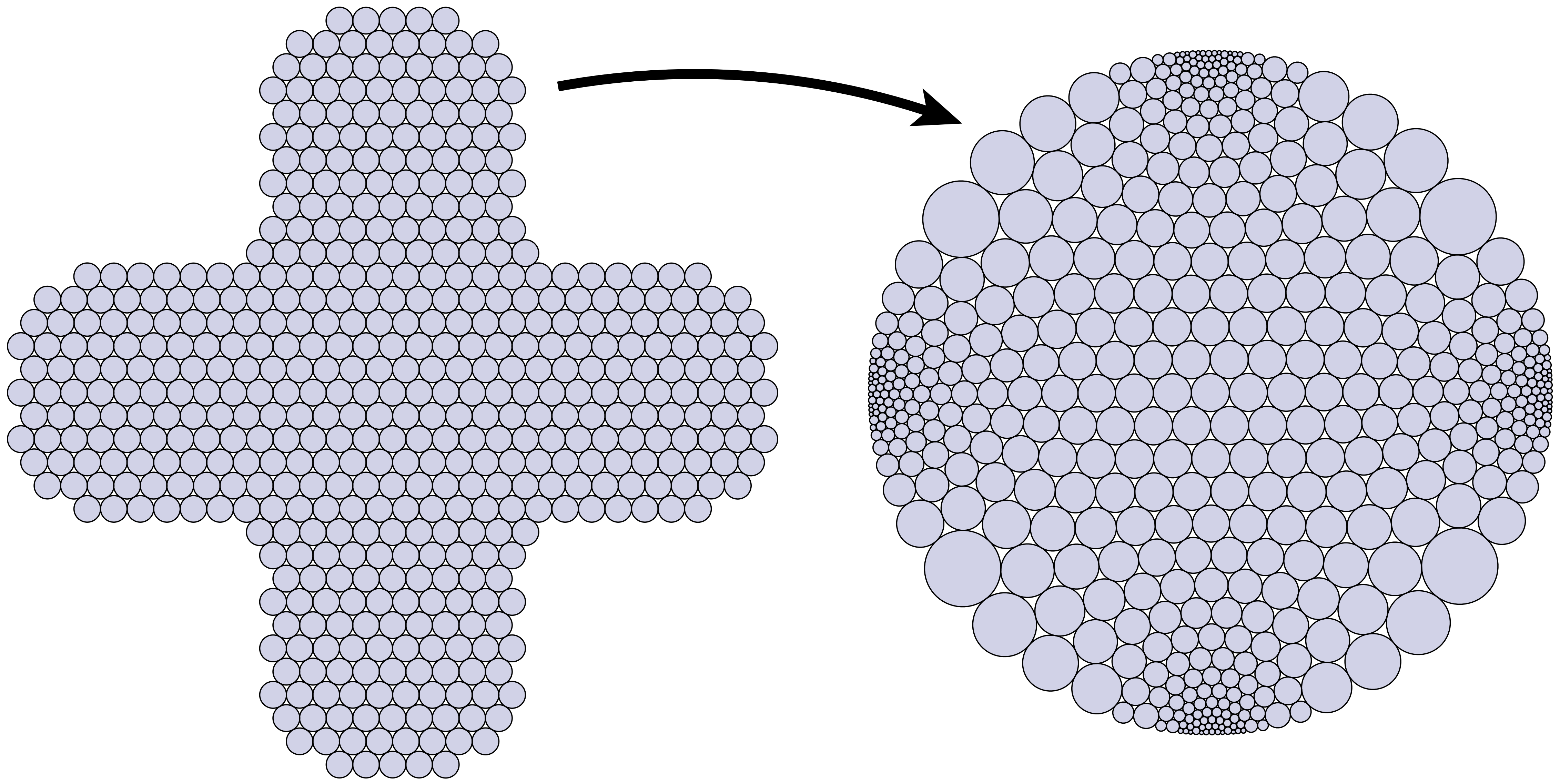
- Circle packings start to look like strong candidate for discretization of conformal maps on simplicial disks:
  - Koebe theorem provides discrete Riemann mapping theorem
  - maps to disk have same symmetry group as in smooth setting (Möbius transformations)
- Major piece still missing: *curvature!*
  - packing based purely on combinatorics ⇒ geometry of domain (i.e., discrete metric) completely ignored
  - disks w/ same combinatorics but very different geometry will be mapped (*i.e.*, packed) in exactly the same way
- $\Longrightarrow$ Circle packings are in general **too flexible**





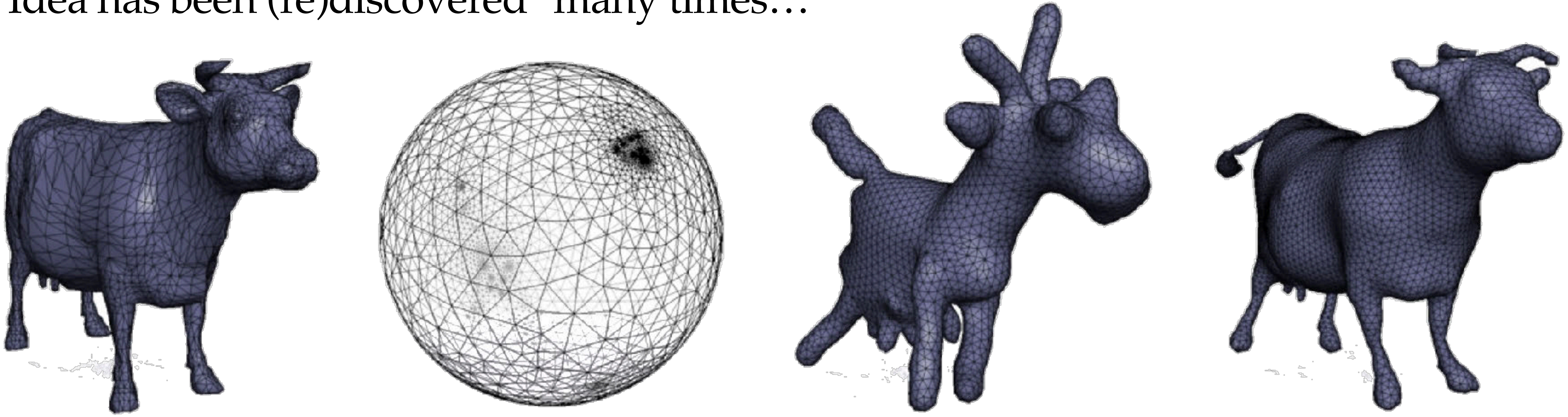
Curvature from Combinatorics

- Why did circle packing work so well for Riemann mapping, but "fails" for conformal flattening?
- Basic insight: **combinatorics encodes curvature**
- Roughly speaking:
  - degree 6 $\iff K = 0$ (*Euclidean*)
  - degree $< 6 \iff K > 0$ (*spherical*)
  - degree $> 6 \iff K < 0$ (*hyperbolic*)
- In 2D, no trouble constructing *initial* packing (or triangulation) that encodes zero curvature: regular hexagons / equilateral triangles



Shape from Combinatorics

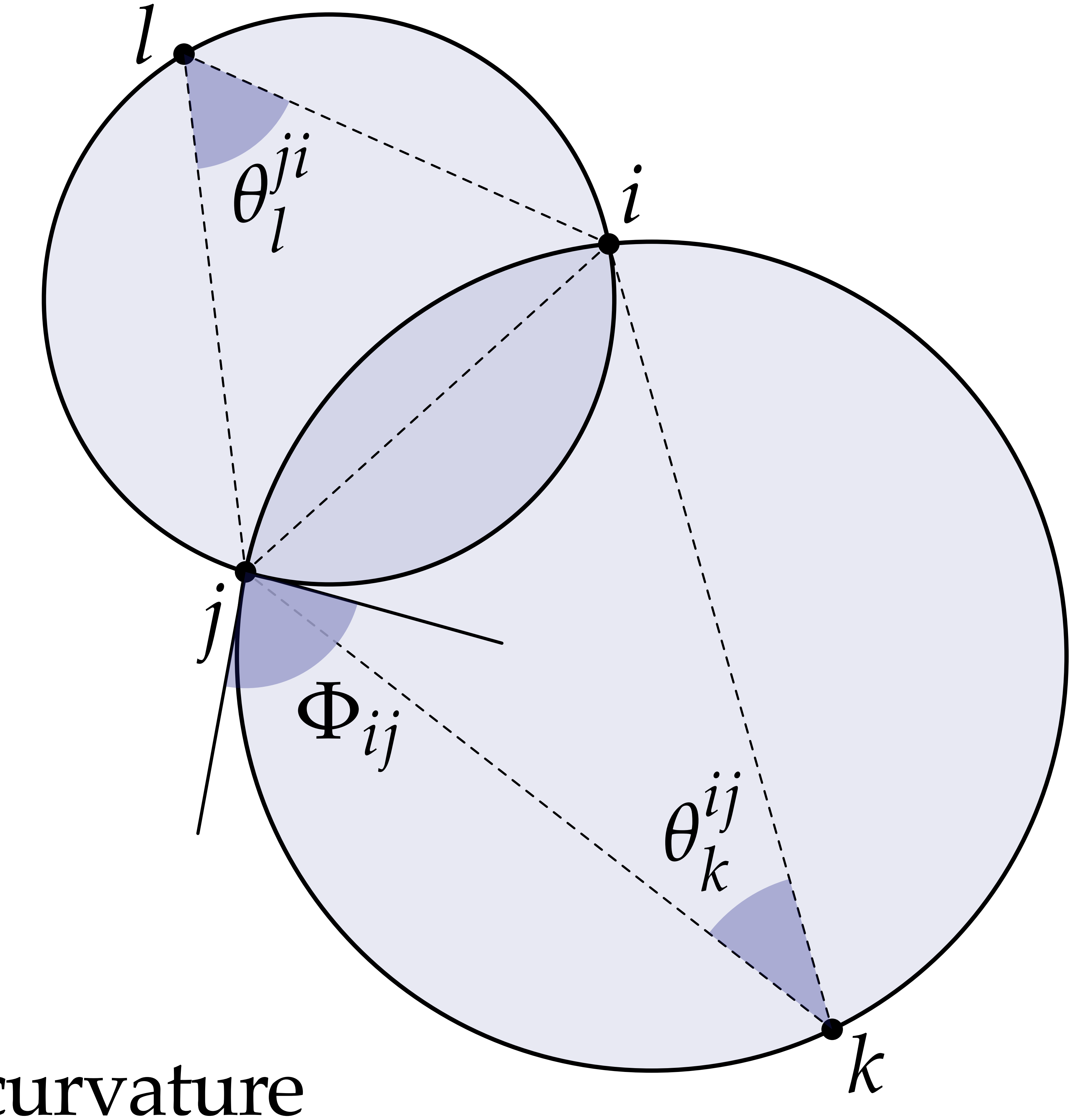
- **Open question:** how can one encode general curved surfaces (not just *constant* curvature) via pure combinatorics?
  - Definition? *Algorithms?*
- Idea has been (re)discovered* many times…



Circle Patterns

- Alternatively, could simply incorporate more geometric information…
- A **circle pattern** (vs. *packing*) associates an angle $\Phi_{ij} \in [0,2\pi)$ to each edge $ij \in E$ of a simplicial disk
- Letting $B \subseteq E$ denote set of boundary edges, the **circle pattern problem** seeks a discrete metric ℓ_{ij} such that

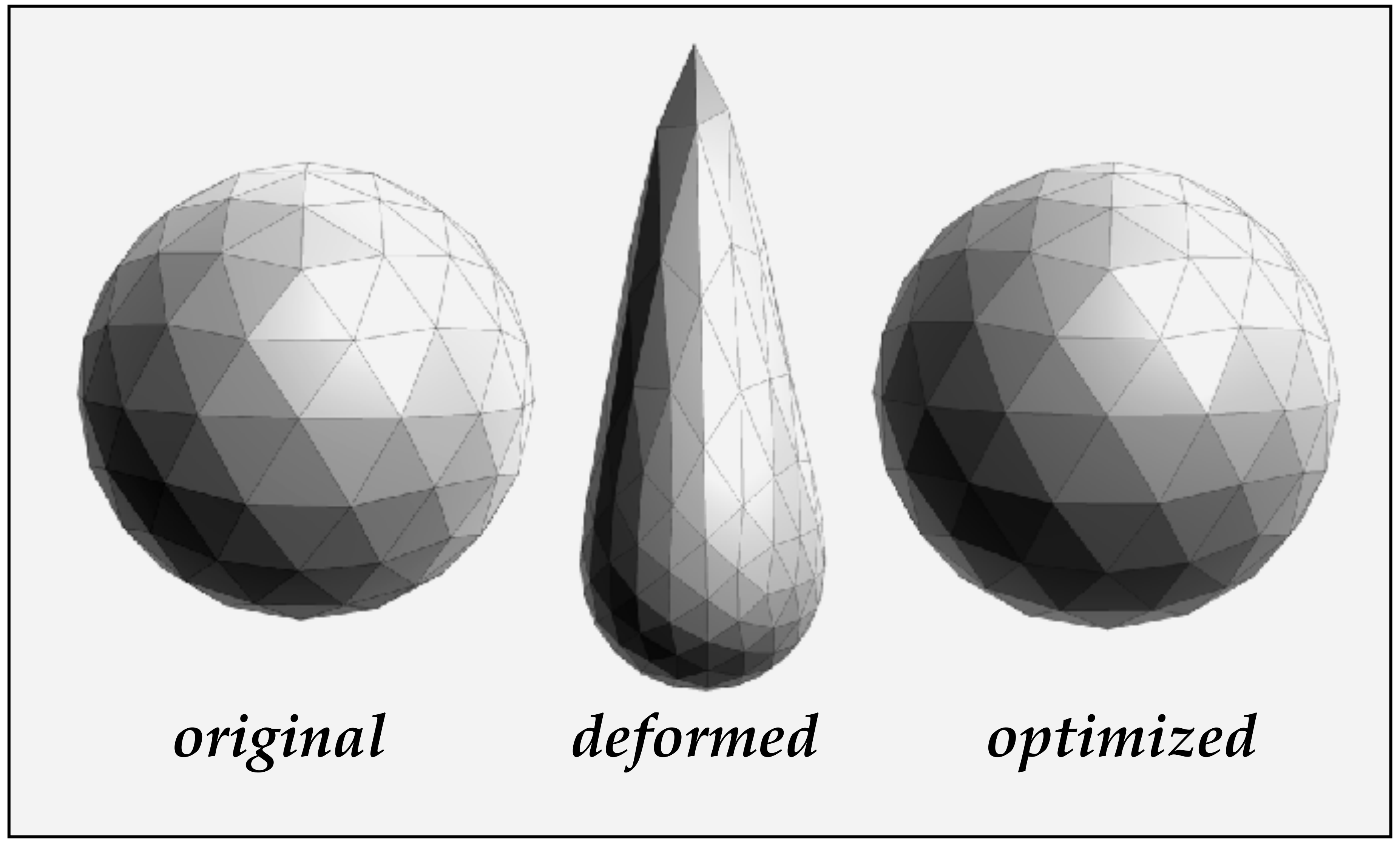
$$\begin{aligned} \Phi_{ij} &= \theta_k^{ij} + \theta_l^{ij}, \quad ij \in E \setminus B \\ \Phi_{ij} &= \theta_k^{ij}, \quad ij \in B \end{aligned}$$

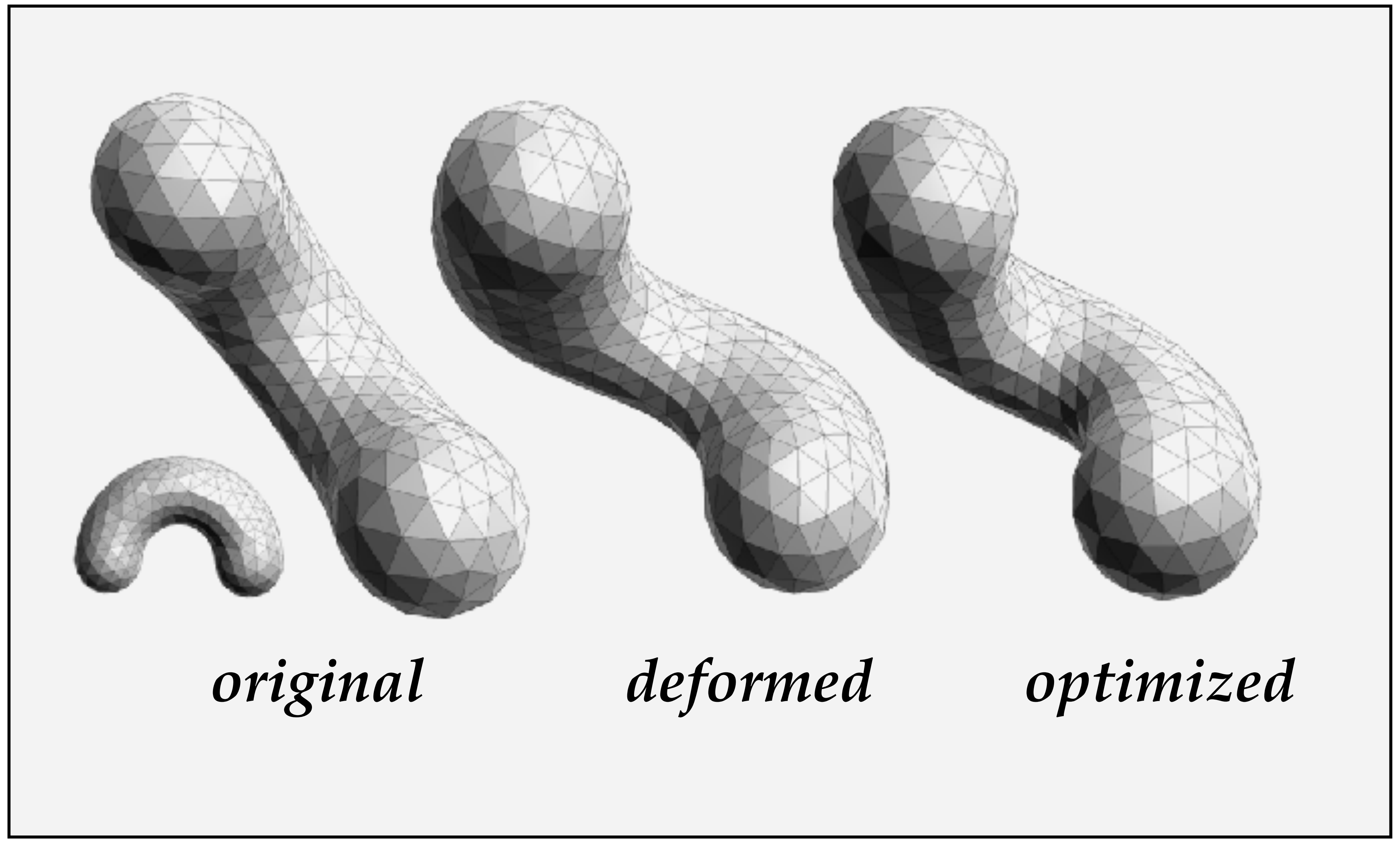


- Angles at edges $ij \in E$ provide control over boundary curvature
- Can be solved (when feasible) by minimizing a convex energy
  - closely connected to variational principles for hyperbolic polyhedra

Discretization via Circle Patterns

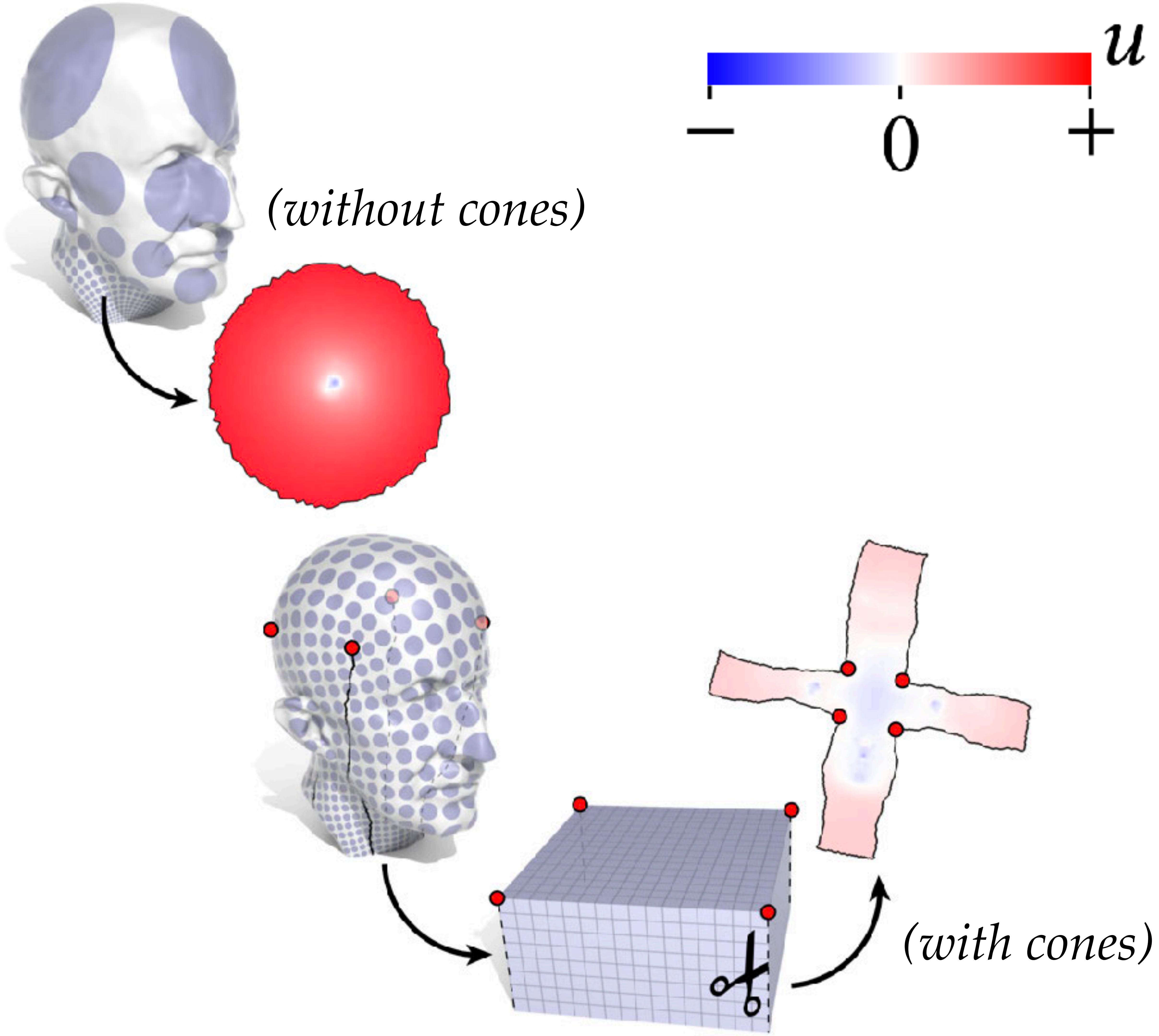
- Despite additional geometric information, circle patterns still do not provide a completely satsifactory discretization:
  - on the one hand can now control angle along boundary
  - however, most simplicial disks cannot be flattened (mapped to the plane) while preserving intersection angles (**too rigid**)
  - numerical experiments suggest they may also be rigid for convex polyhedra in 3-space (but not nonconvex?)





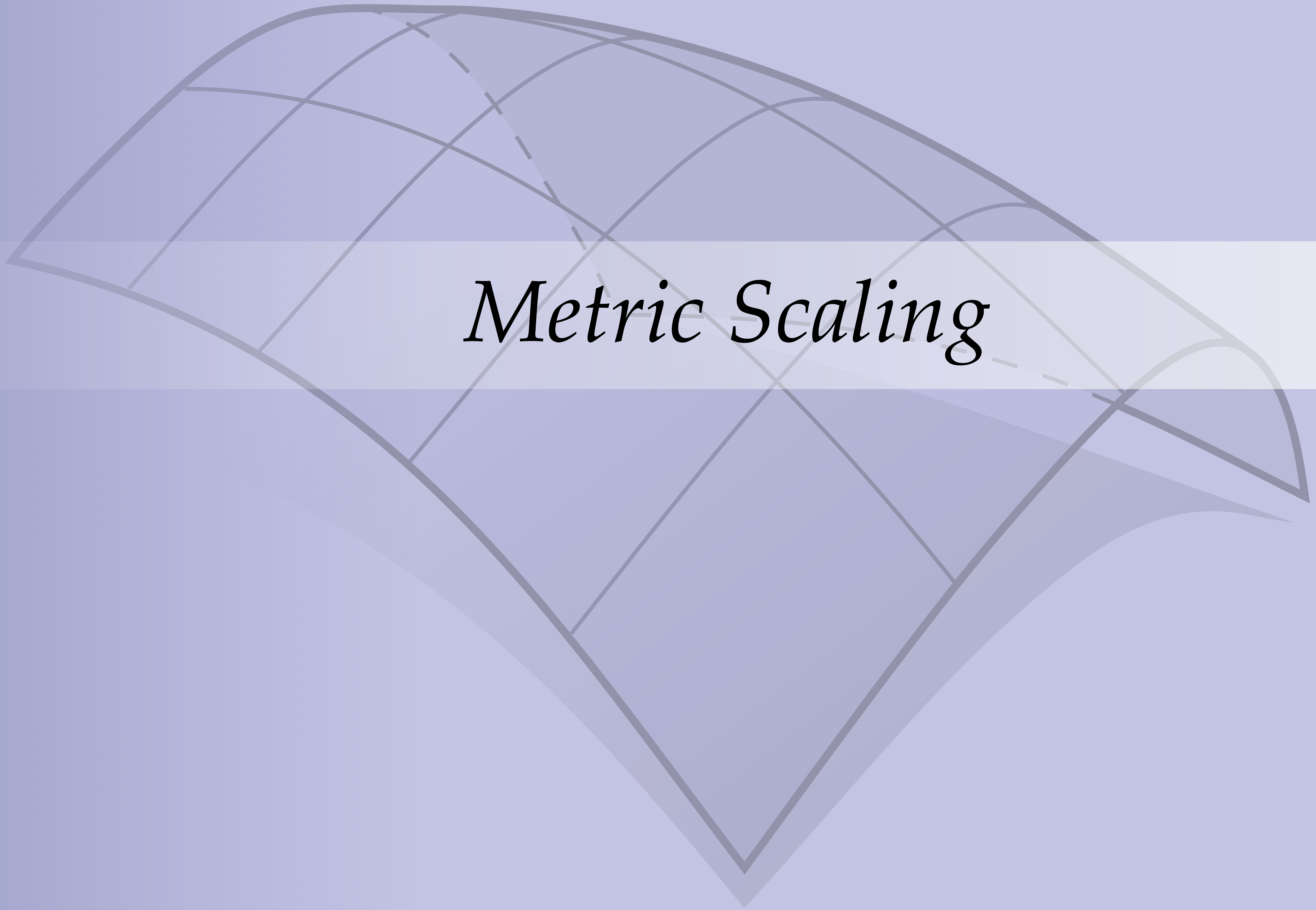
Circle Patterns

- Despite rigidity, circle patterns can still be used for practical algorithms
- As with angles, holomorphic functions, find flattening with minimal deviation of intersection angles (in L^2 sense)
- Naturally allows incorporation of *cone singularities*
  - Rather than flatten directly to plane, flatten to metric w/ isolated points of curvature; then cut & flatten
  - Significantly reduce distortion of *area*



KHAREVYCH, SPRINGBORN, SCHRÖDER, *"Discrete Conformal Mappings via Circle Patterns"* (2006)

Metric Scaling



Conformally Equivalence of Riemannian Metrics

- Two Riemannian metrics g_1, g_2 on a manifold are **conformally equivalent** if they are related by a positive scaling at each point:

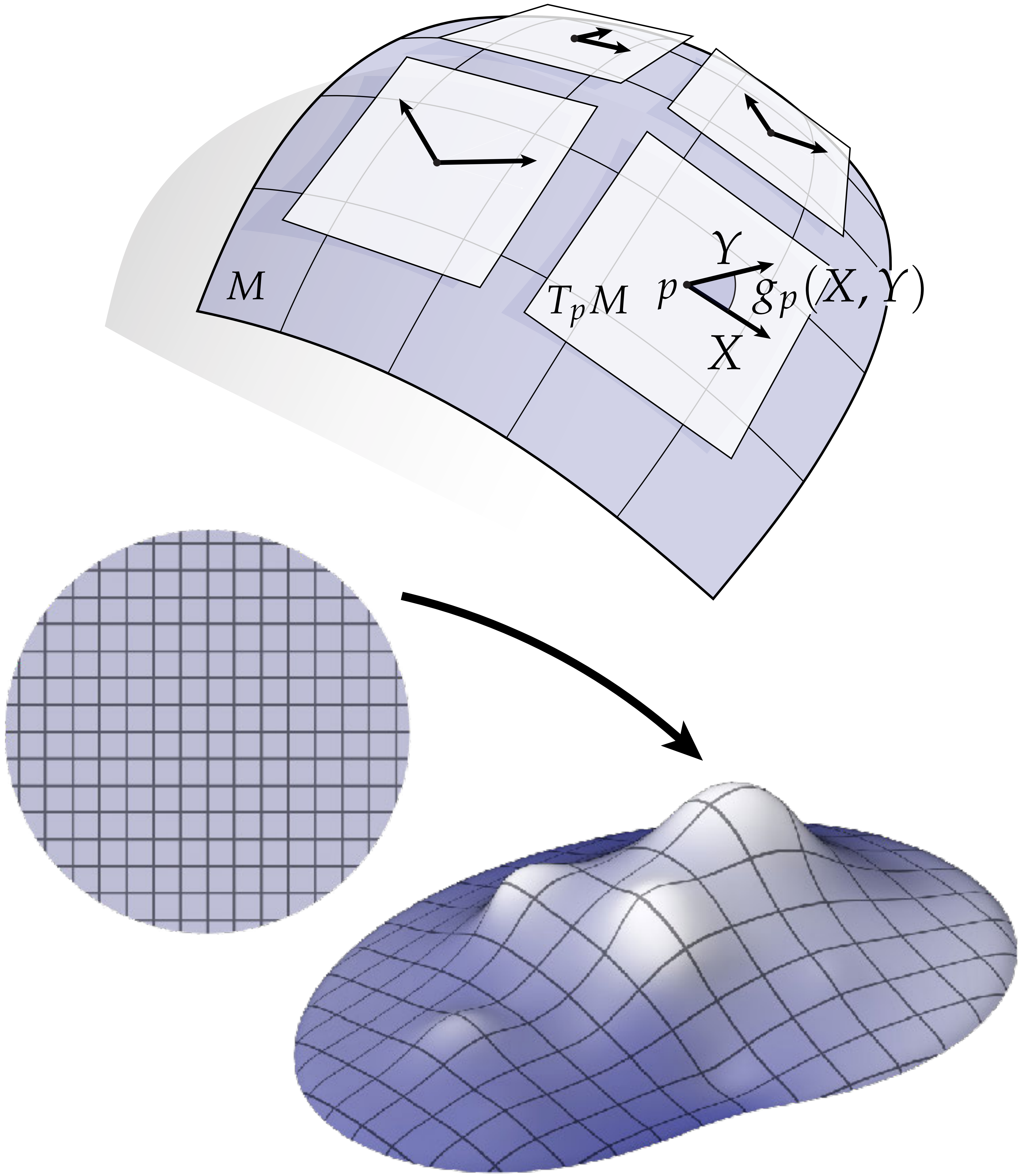
$$g_2 = e^{2u} g_1, \quad u : M \to \mathbb{R}$$

- **Idea:** define conformal flattening of a simplicial disk as conformally equivalent discrete metric of zero Gaussian curvature

- Need to define:

1. discrete Gaussian curvature

2. conformal equivalence of discrete metrics



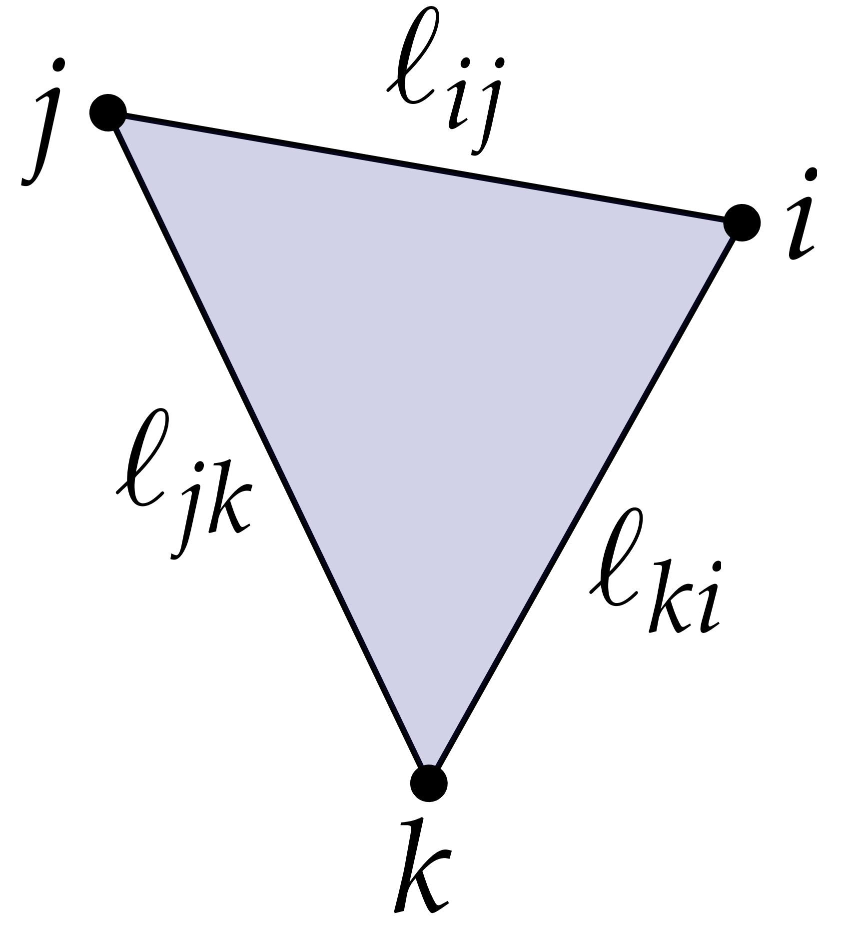
Discrete Metric

Recall that a **discrete metric** on an abstract simplicial surface $K = (V, E, F)$ is simply an assignment of edge lengths

$$\ell_{ij} : E \to \mathbb{R}_{>0}$$

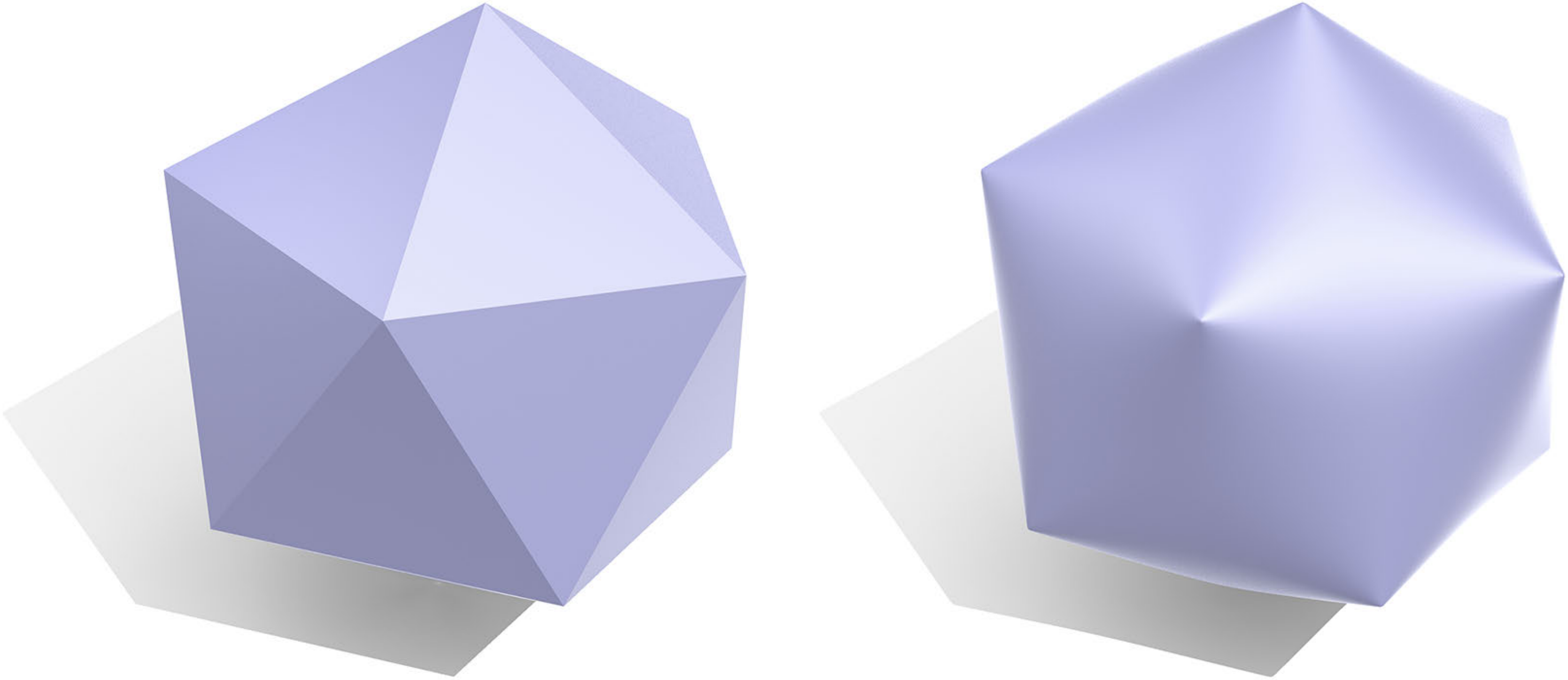
satisfying the triangle inequality in each face, *i.e.*,

$$\ell_{ij} + \ell_{jk} \geq \ell_{ik}, \quad \forall ijk \in F.$$

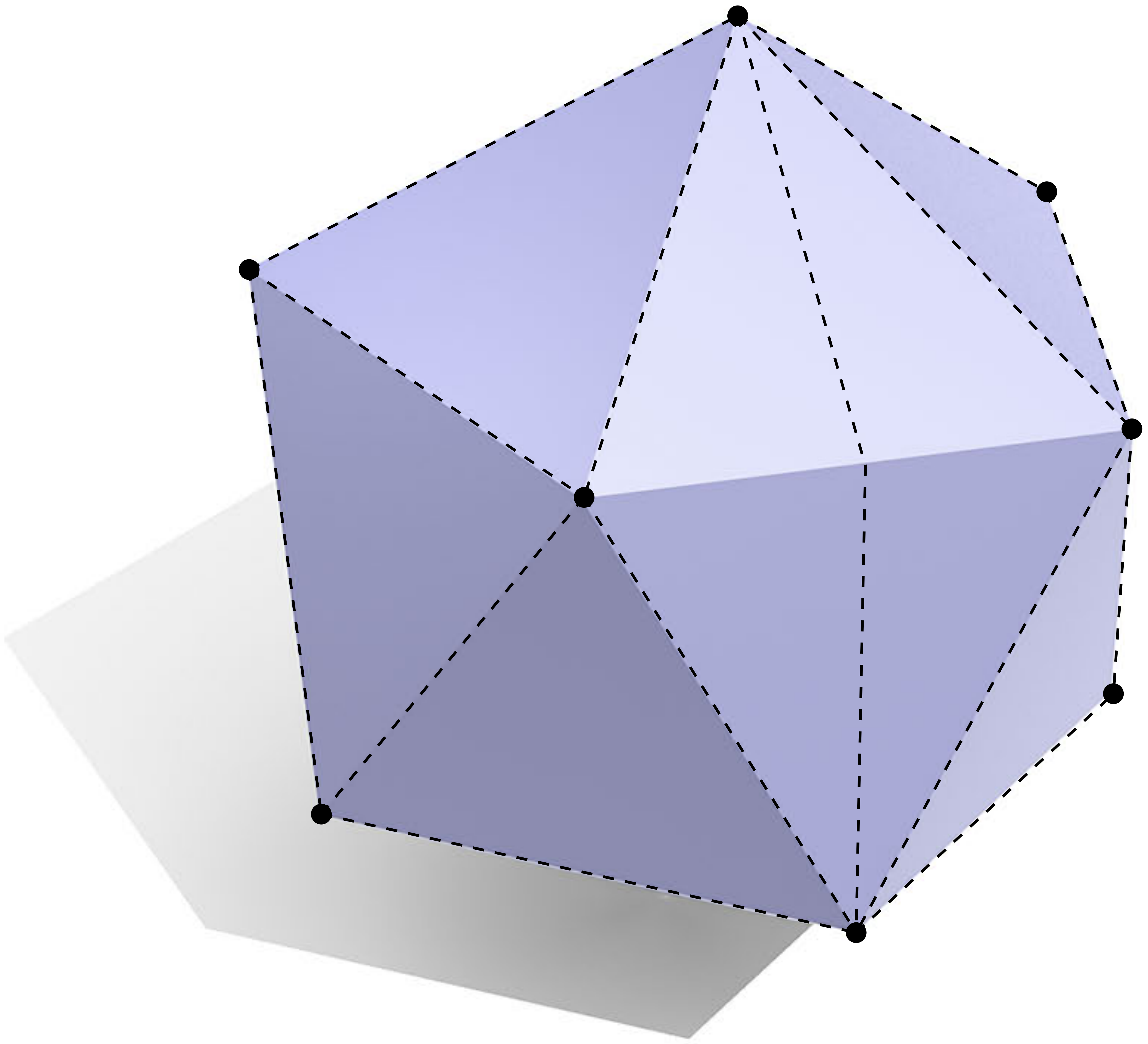


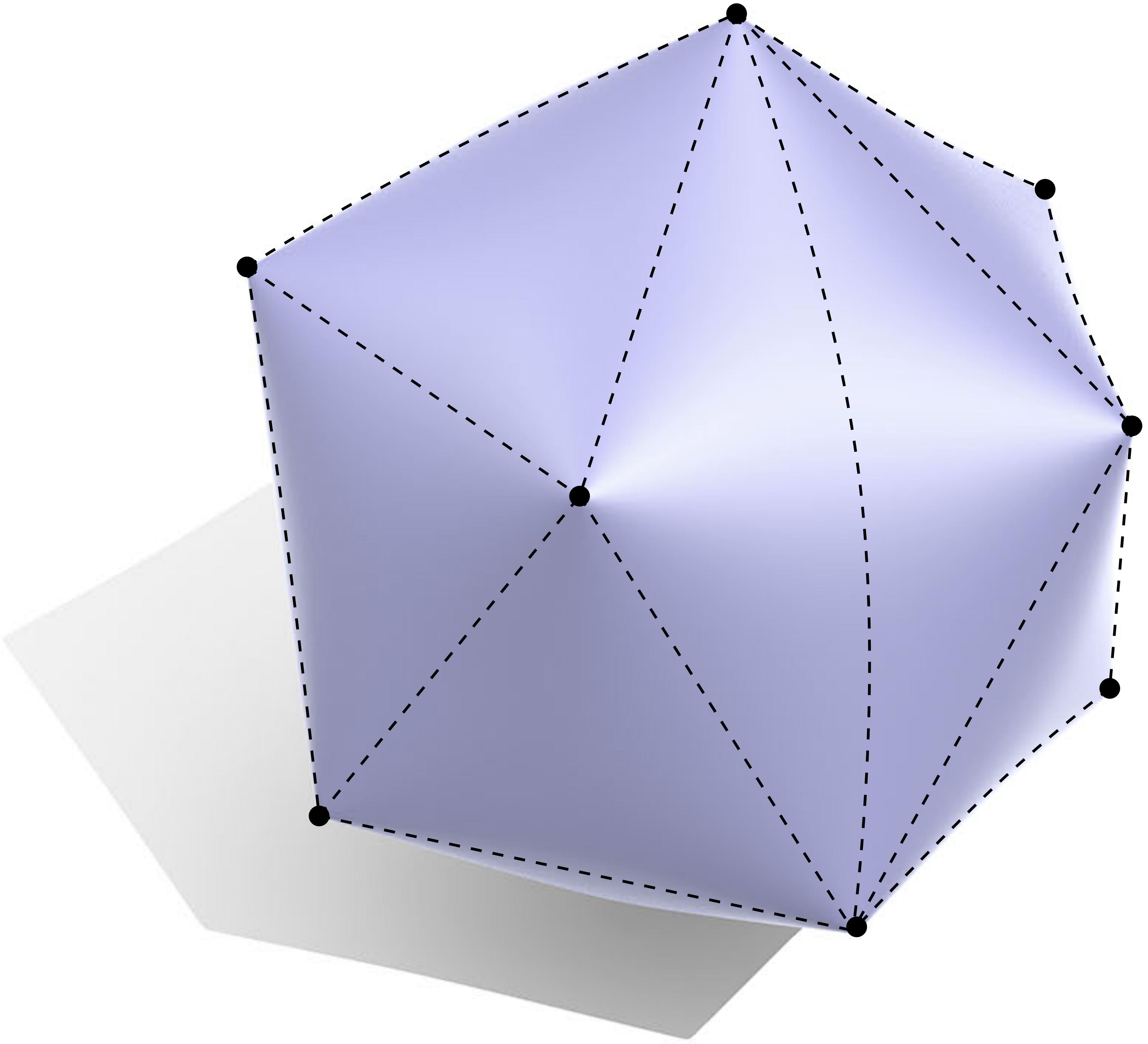
- Naturally associated to a piecewise Euclidean metric obtained by gluing together Euclidean triangles (of prescribed length) along shared edges.
- Result is a or *cone metric*: Gaussian curvature is nonzero only at isolated *cone points* (corresponding to vertices); in this sense, edges are superficial.

Discrete Metric — Visualized



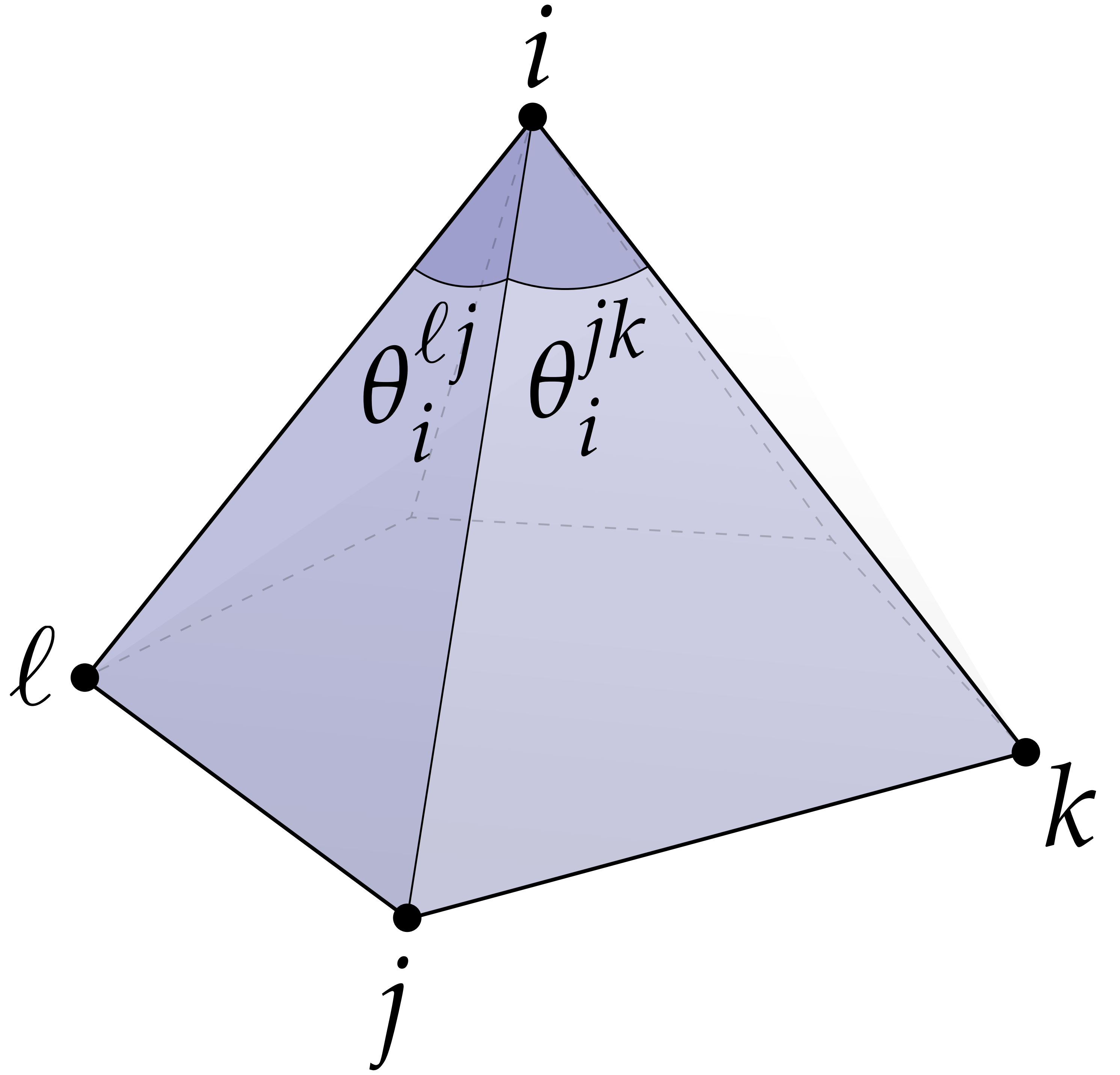
Discrete Metric — Visualized



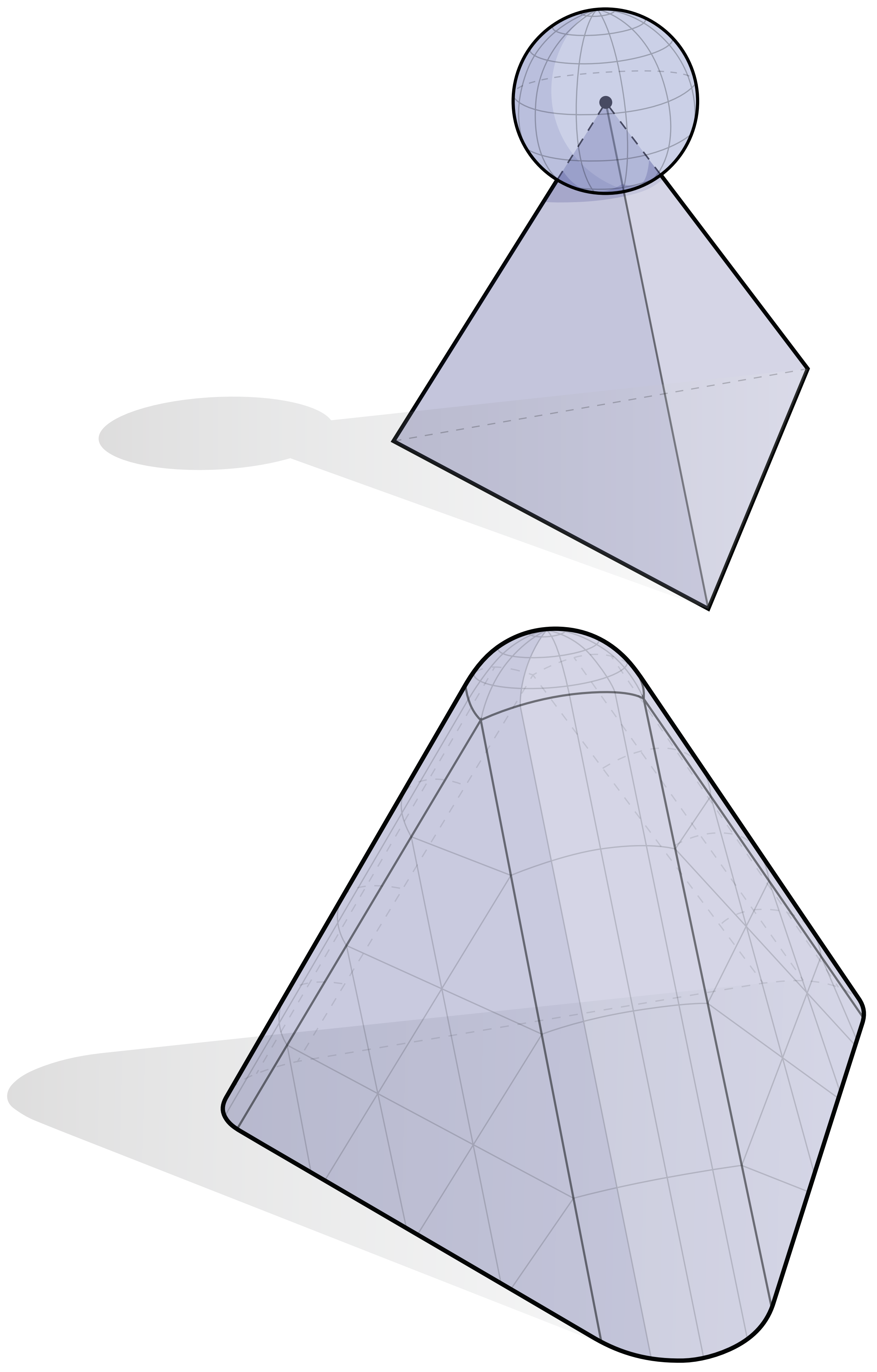


Discrete Gaussian Curvature

Discrete Gaussian curvature of a vertex i is equal to the **angle defect**, *i.e.*, the deviation of interior angles around the vertex from the Euclidean angle sum 2π:



$$K_i := 2\pi - \sum_{ipq} \theta_i^{pq}$$



(Angles easily obtained from discrete metric via cosine or half-angle formula.)

Conformal Equivalence of Discrete Metrics

Definition. Two discrete metrics $\ell, \tilde{\ell} : E \to \mathbb{R}_{>0}$ on a simplicial surface $K = (V, E, F)$ are **discretely conformally equivalent** if there exists an assignment of *log conformal factors* $u_i \in \mathbb{R}$ to each vertex $i \in V$ such that

$$\tilde{\ell}_{ij} = e^{(u_i+u_j)/2}\ell_{ij} \qquad \forall ij \in E$$

- Initially looks like we are just "aping" the smooth definition

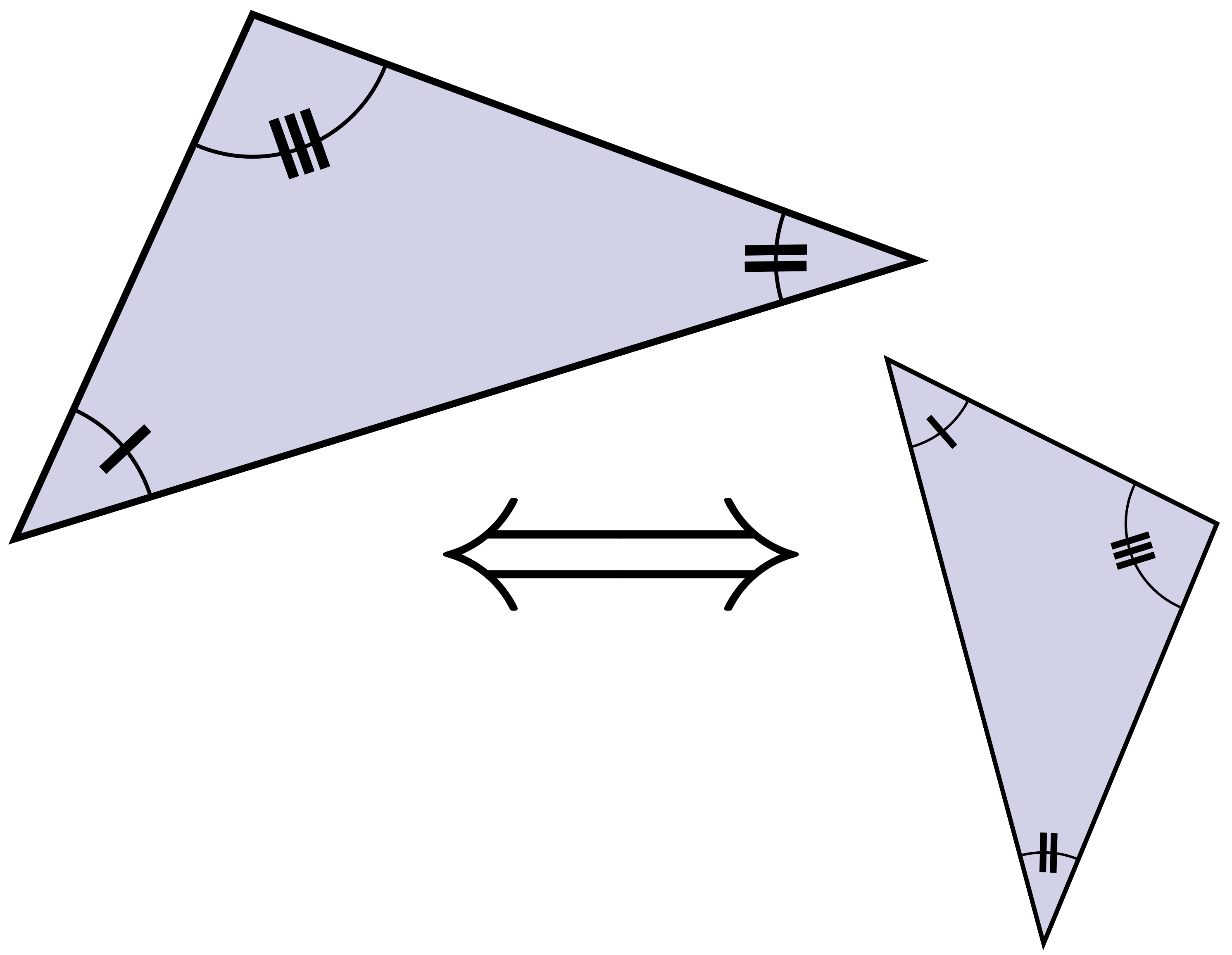
$$\tilde{g} = e^u g$$

- This notion of discrete conformal equivalence will turn out to provide exactly the right amount of flexibility, *i.e.*, it is neither "too rigid" nor "too flexible," but rather "just right."

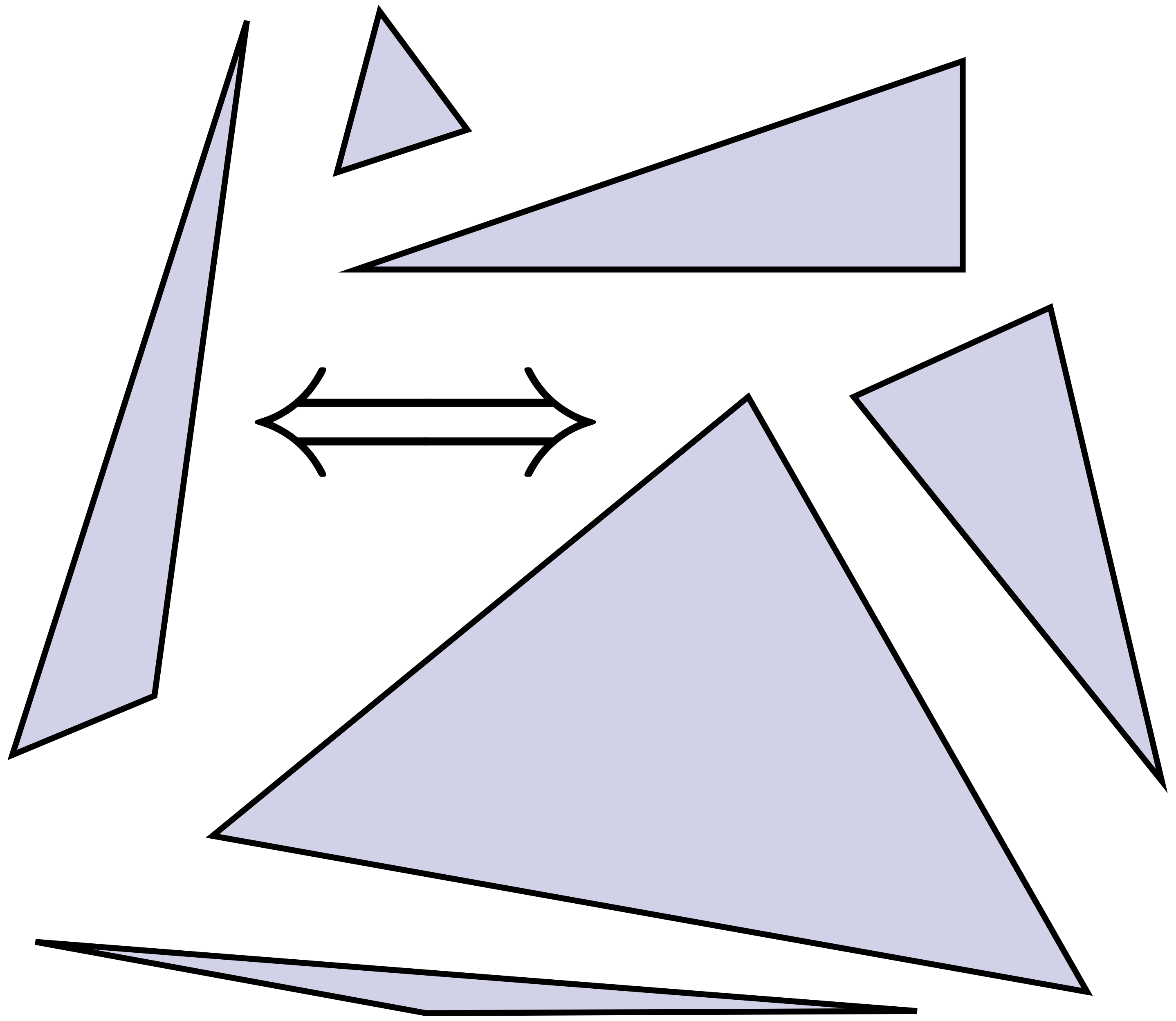
Flexibility of Triangles

- For a *single* triangle, equivalence classes with respect to preservation of interior angles are just similarities (too rigid). What about discrete conformal equivalence?

Proposition. Any two discrete metrics $\ell, \tilde{\ell}$ on a single triangle $ijk \in F$ are discretely conformally equivalent.



INTERIOR ANGLES



DISCRETE METRICS

Flexibility of Triangles (Proof)

Proof. We seek log conformal factors u_i, u_j, u_k such that

$$e^{(u_a+u_b)/2} = \tilde{\ell}_{ab}/\ell_{ab} \tag{1}$$

for all $ab \in ijk$. Let $\lambda_{ij} := 2\log(\ell_{ij})$, and similarly for $\tilde{\ell}$. Then by taking the logarithm of (1) we get a linear system

$$u_a + u_b = \tilde{\lambda}_{ab} - \lambda_{ab} \qquad \forall ab \in ijk.$$

This system has a unique solution, given by

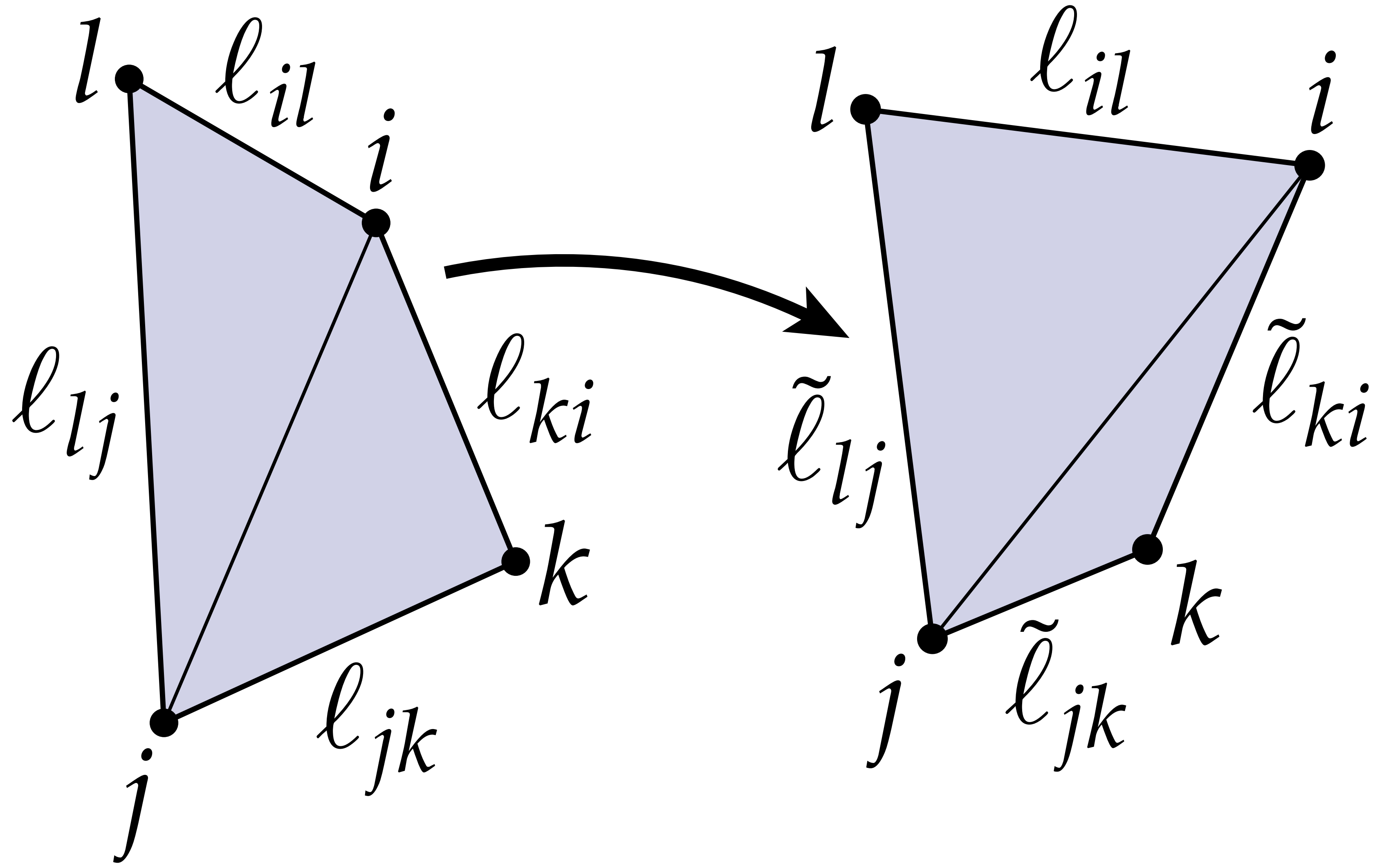
$$e^{u_i} = \frac{\tilde{\ell}_{ij}}{\ell_{ij}} \frac{\ell_{jk}}{\tilde{\ell}_{jk}} \frac{\tilde{\ell}_{ij}}{\ell_{ki}}.$$

(and similarly for u_j, u_k).

Length Cross Ratio

- Is discrete conformal equivalence *too* flexible? *All* triangles are equivalent!
- However, scale factors are shared by multiple triangles…
- Consider the **length cross ratio** associated with any interior edge *ij*:

$$\mathfrak{c}_{ij} := \frac{\ell_{il}\ell_{jk}}{\ell_{ki}\ell_{lj}}$$



Theorem. Two discrete metrics $\ell, \tilde{\ell}$ are conformally equivalent if and only if they induce the same length cross ratios; *i.e.*, if for all $ij \in E \setminus B$,

$$\mathfrak{c}_{ij} = \tilde{\mathfrak{c}}_{ij}.$$

Length Cross Ratio (Proof)

Proof. Suppose that ℓ and $\tilde{\ell}$ are conformally equivalent. Then for all edges $ij \in E$,

$$\tilde{\mathfrak{c}}_{ij} = \frac{\tilde{\ell}_{il}\tilde{\ell}_{jk}}{\tilde{\ell}_{ki}\tilde{\ell}_{lj}} = \frac{e^{(u_i+u_l)/2}e^{(u_j+u_k)/2}\ell_{il}\ell_{jk}}{e^{(u_k+u_i)/2}e^{(u_l+u_j)/2}\ell_{ki}\ell_{lj}} = \frac{\ell_{il}\ell_{jk}}{\ell_{ki}\ell_{lj}} = \mathfrak{c}_{ij},$$

i.e., the scale factors simply cancel.

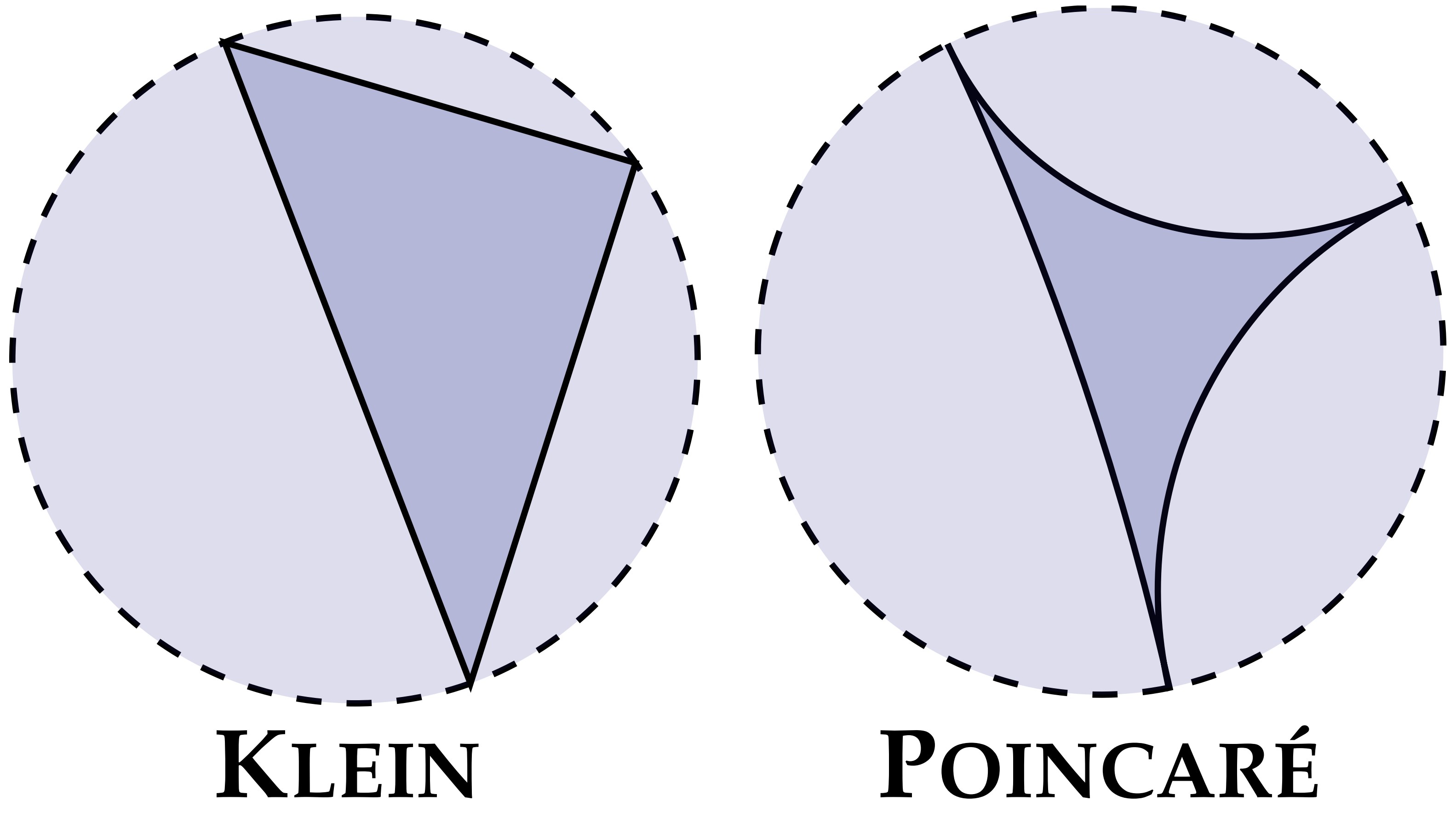
Now suppose that the cross ratios are equal. For any two adjacent triangles ijk, jil, we can find log scale factors u_i, u_j, u_k and v_j, v_i, v_l expressing equivalence between the old and new metric (independently for each triangle) using the expression from the previous proposition. Compatibility of these factors on shared vertices yields

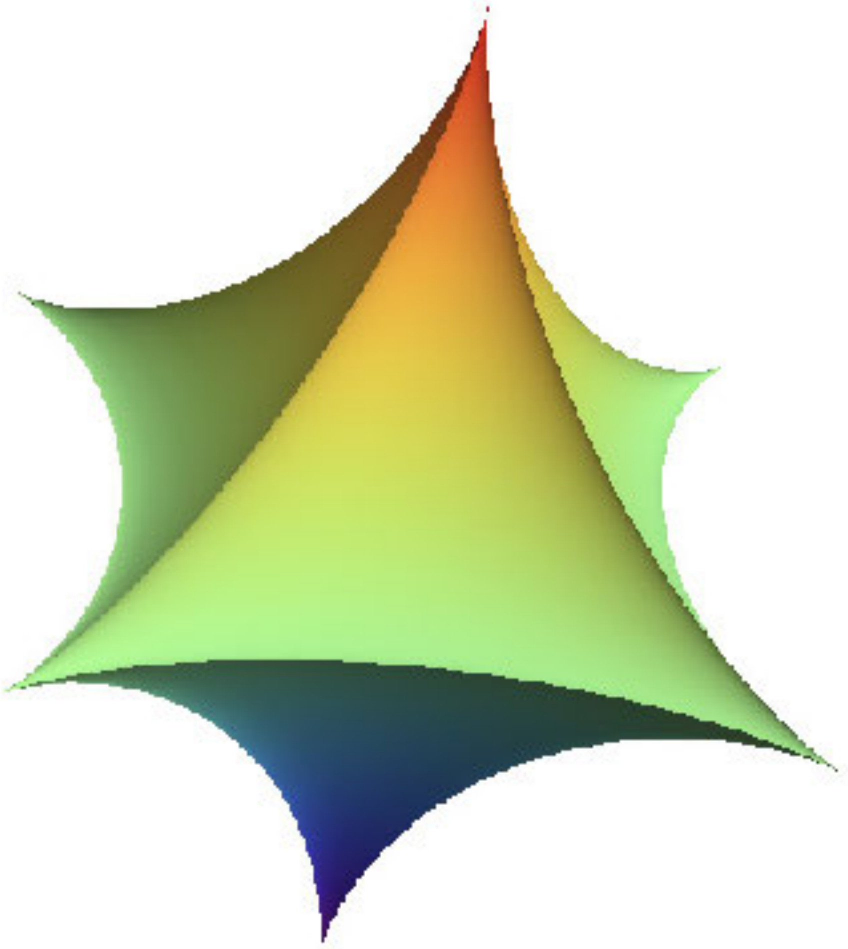
$$\frac{\ell_{jk}}{\tilde{\ell}_{jk}}\frac{\tilde{\ell}_{ki}}{\ell_{ki}} = \frac{\tilde{\ell}_{il}}{\ell_{il}}\frac{\ell_{lj}}{\tilde{\ell}_{lj}},$$

i.e., equivalence of cross ratios.

Discrete Conformal Equivalence

- From here, can develop a rich theory of discrete conformal equivalence mirroring many properties found in the smooth setting*
- *E.g.*, cross ratios are preserved by Möbius transformations
- Also have "discrete Teichmüller spaces":
  - discrete metrics : $|E|$-dimensional
  - conformal rescalings: $|V|$-dimensional
  - discrete equivalence classes: $|E| - |V| = 6g - 6 + 2|V|$
  - same as Teichmüller space of genus g w/ $|V|$ punctures
- Connection to hyperbolic polyhedra:
  - each Euclidean triangle is ideal hyperbolic triangle in Klein model
  - discrete conformal equivalence $\Longleftrightarrow$ isometries of hyperbolic polyhedron





*BOBENKO, PINKALL, SPRINGBORN, *"Discrete conformal maps and ideal hyperbolic polyhedra"* (2010)

Discrete Uniformization

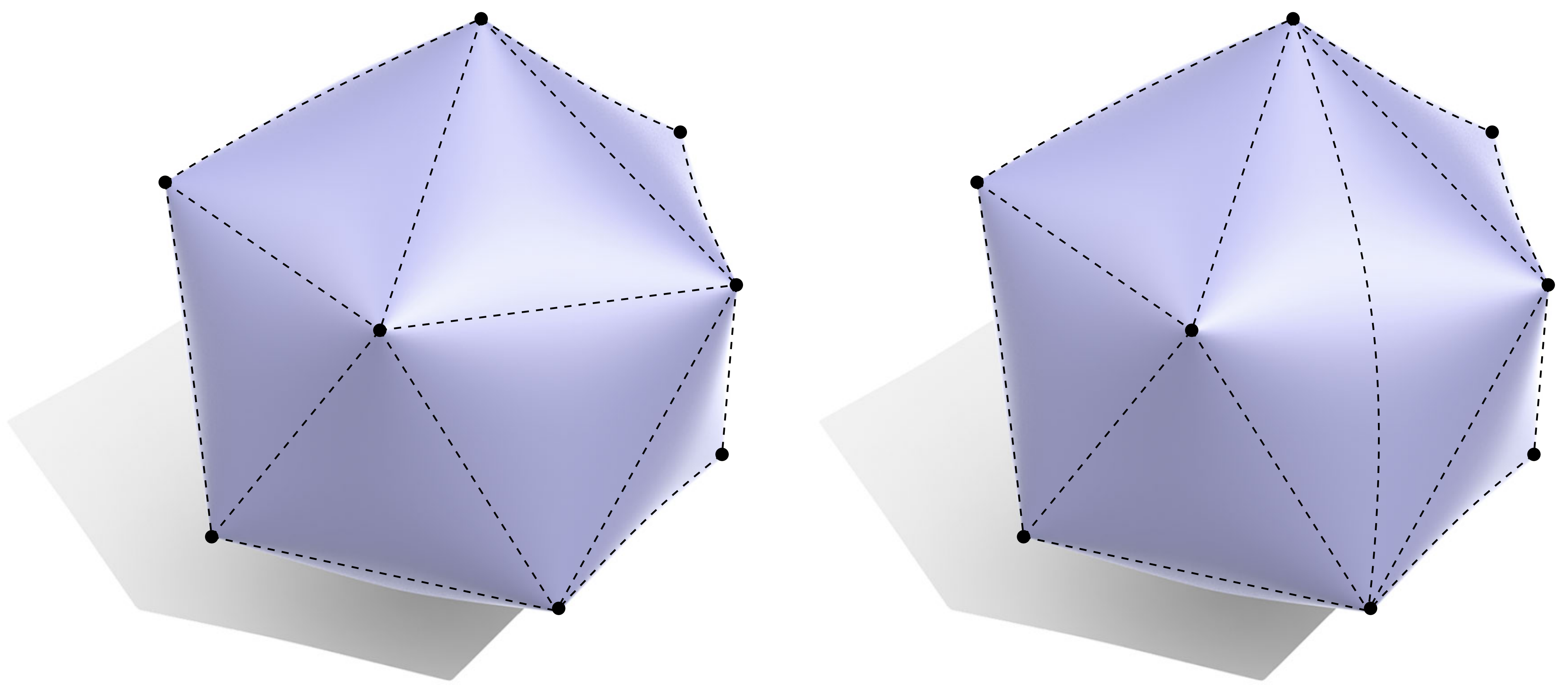
- Still haven't resolved original problem: how do we define (and ultimately, construct) a conformal flattening of a simplicial disk?
- Special case of *uniformization*: for a given Riemannian metric, find a conformally equivalent metric of constant curvature.
- On smooth surfaces, key tool is *Yamabe flow* (2D Ricci flow)
- For discrete metric, can formulate a very simple discrete Yamabe flow*: at each vertex $i \in V$, change in log scale factor u_i is proportional to difference between target angle sum Θ_i and current angle sum:

$$\frac{d}{dt} u_i = -\frac{1}{2}\left(\Theta_i - \sum_{ijk} \theta_i^{jk}\right)$$

*Luo, *"Combinatorial Yamabe Flow on Surfaces"* (2004)

Discrete Uniformization — Existence

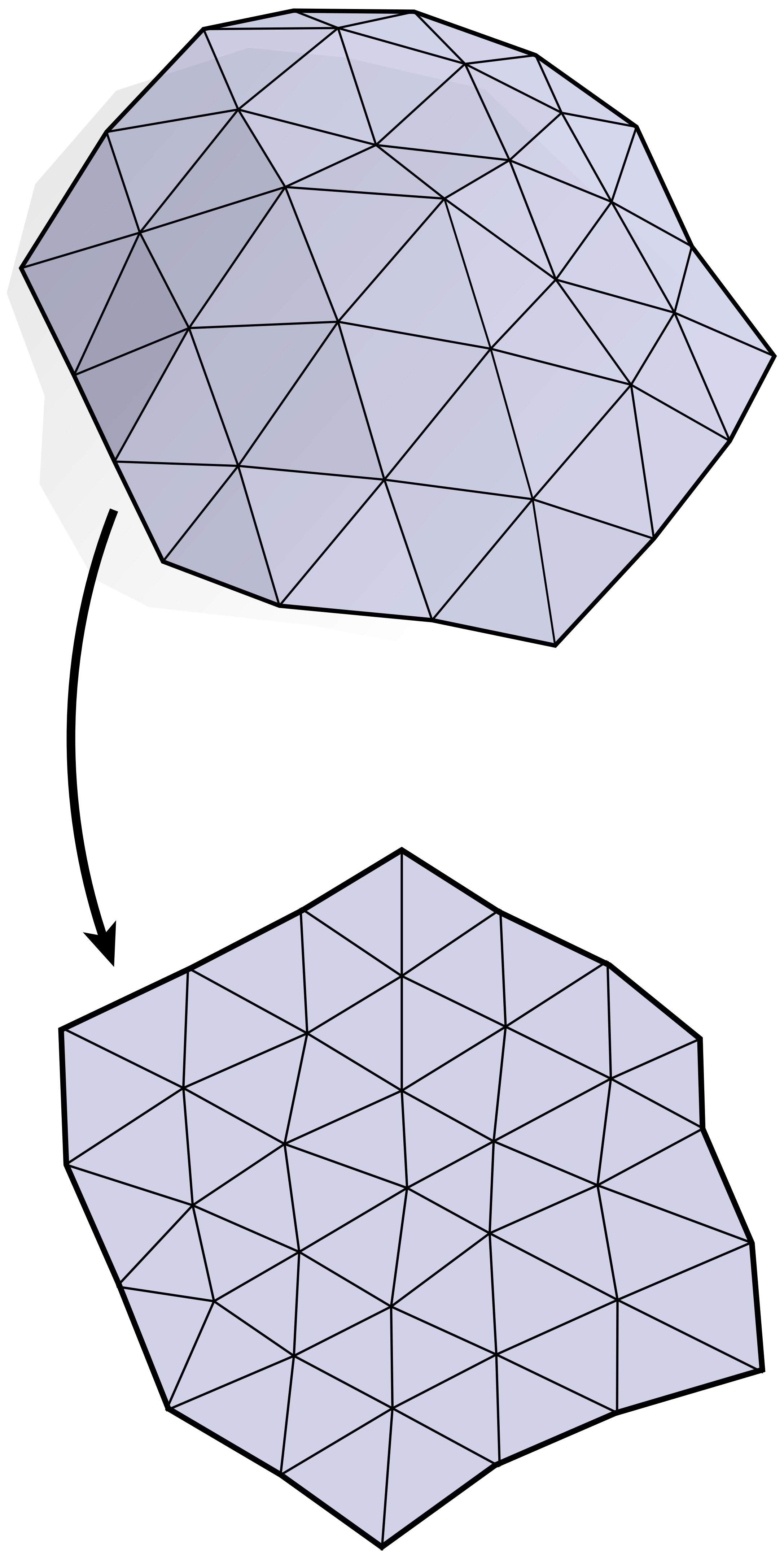
- **Key question:** can one *always* find a conformally equivalent discrete metric with prescribed Gaussian curvature (*i.e.*, angle defect)?
- Some key results:
  - (Luo 2004) *Conjecture:* Discrete Yamabe flow will uniformize, w/ sufficient "surgery"
  - (Springborn, Schröder, Pinkall 2008) Discrete Yamabe flow comes from convex variational principle; use to perform Newton descent (w/ cotan-Laplace as Hessian!)
  - (Bobenko, Pinkall, Sprinborn 2010) Connection to variational principles for hyperbolic polyhedra
  - (Gu, Guo, Luo, Sun, Wu 2013 & 2014) Existence in hyperbolic ($g < 0$), toroidal case ($g = 0$)
  - (Springborn 2017) Existence in spherical case
- **Key insight**: edge flips needed to ensure existence!
  - Euclidean, or *Ptolemy* (hyperbolic)
- More than enough to solve our original problem…!



Summary

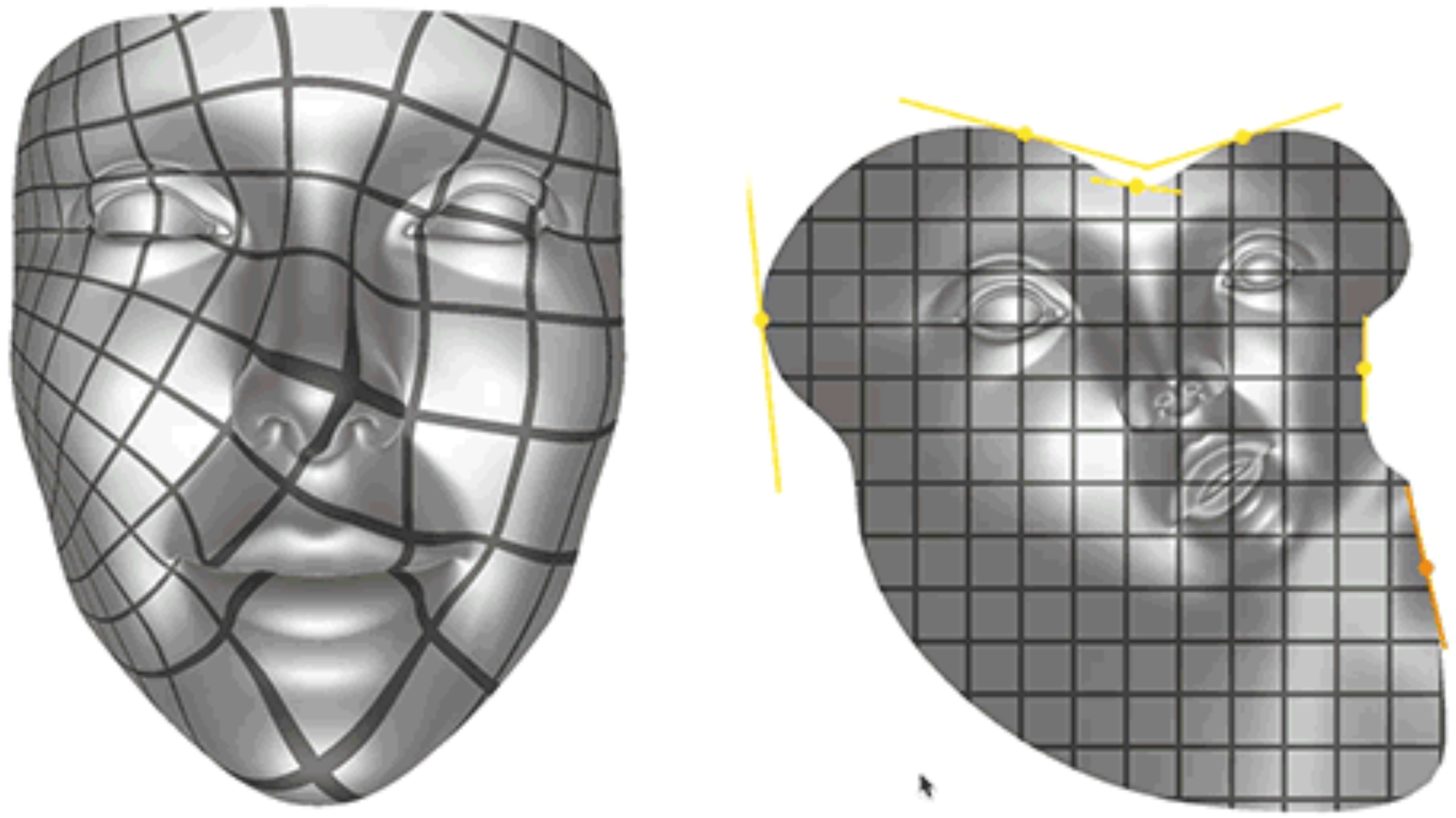
Conformal Flattening of a Discrete Disk

- Considered several characterizations:
  - **angle preservation** — *too rigid*
  - **piecewise holomorphic** — *too rigid*
  - **circle preservation** — *too flexible*
  - **metric scaling** — *just right*
- Several approaches we didn't cover (see notes)
  - **harmonic maps**
  - **Hodge star operator**
  - **conjugate harmonic functions**
  - …
- None seem to provide theory as rich as metric scaling…



Boundary First Flattening

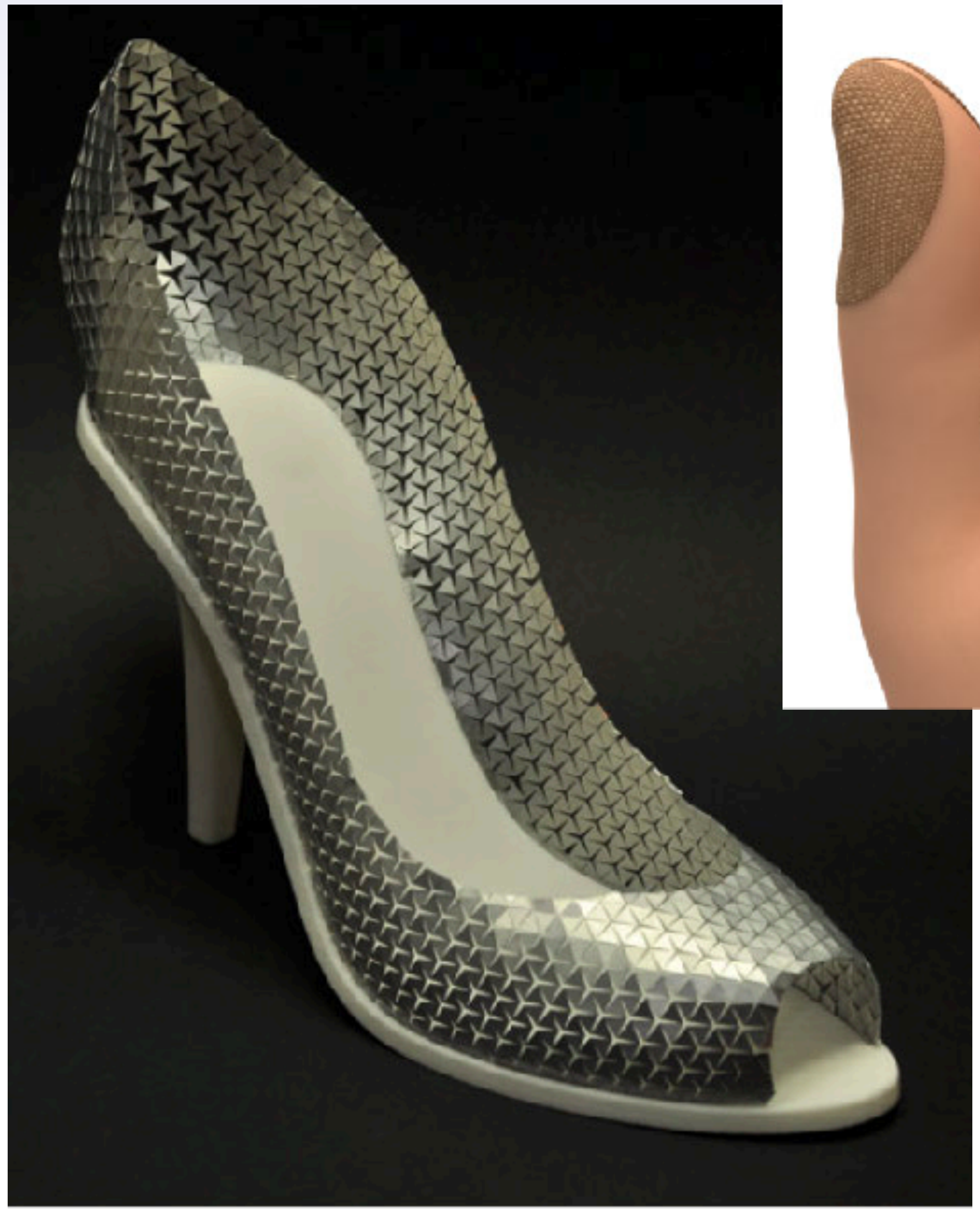
- Recent approach* does however provide exactly the right degree of flexibility
- **Basic idea**:
  - *first* compute map along boundary, *then* interpolate via conjugate harmonic coordinate functions
  - boundary data obtained by operators that are linear in discrete setting:
    - *Poincaré-Steklov operators*
      - *(Hilbert transform, Dirichlet-to-Neumann map)*
    - *discrete Cherrier problem*
- Since based on linear equations, very "practical" (~100x faster than Yamabe; no edge flips)

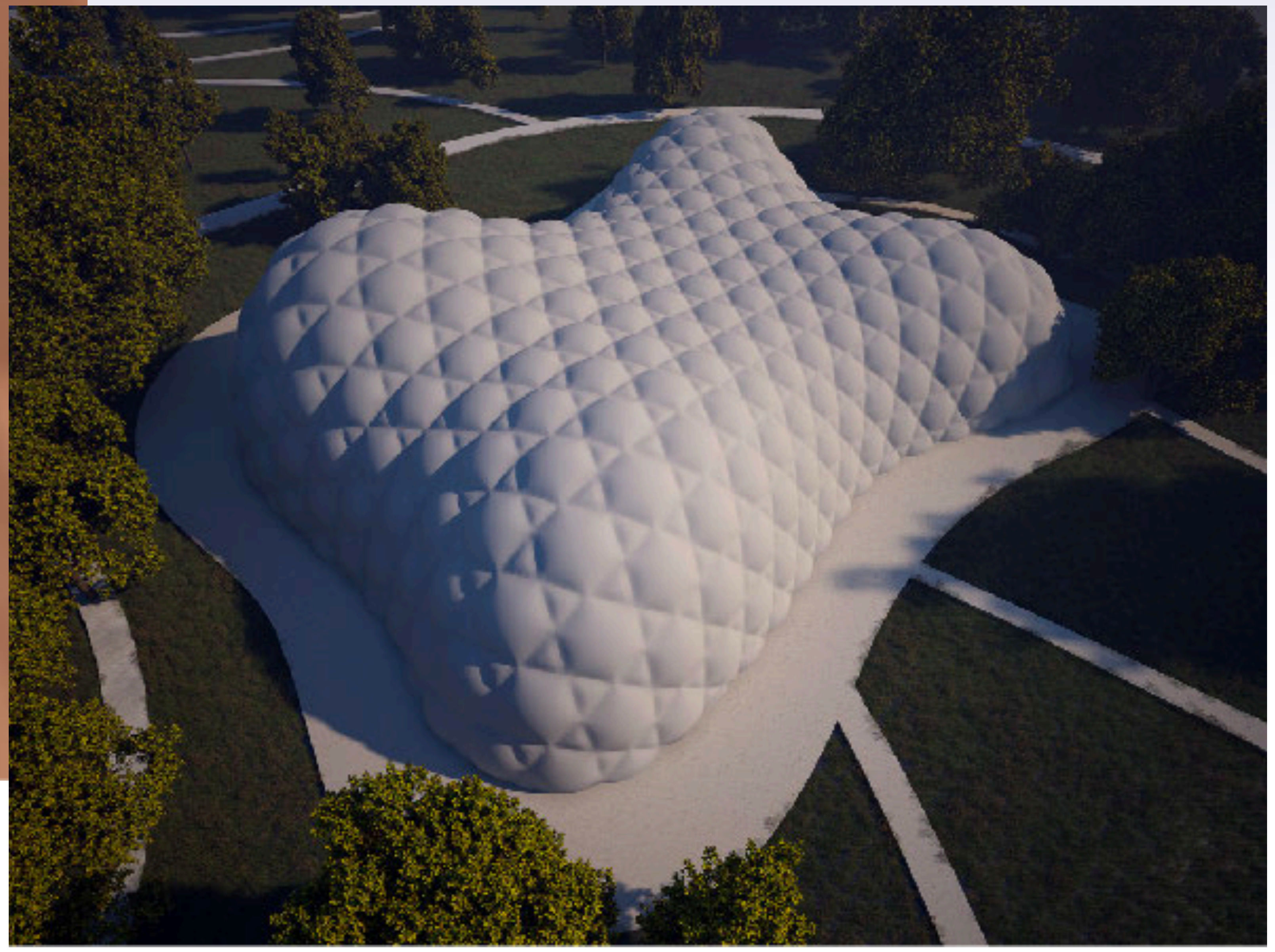


[DEMO]

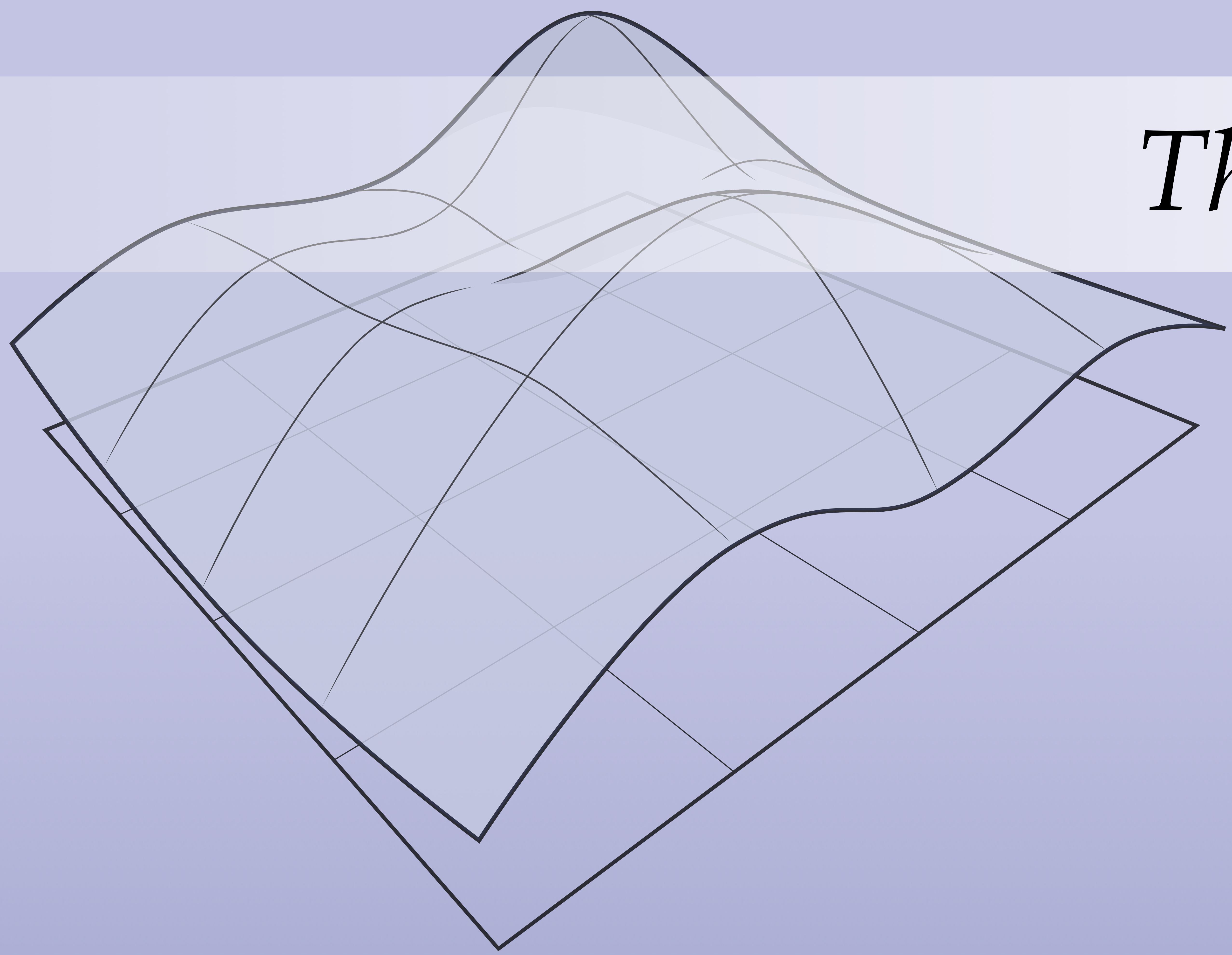
*Sawhney & Crane, *"Boundary First Flattening"* (2017)

Next up: Applications





Thanks!



AMS SHORT COURSE

DISCRETE DIFFERENTIAL GEOMETRY

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